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Sound Propagation in a Liquid Overlying a Viscoelastic Halfspace

Allen H. Magnuson
Mechanical Engineering Department
The University of New Hampshire

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Cooperative University-Industry Research Project
between

University of New Hampshire
Durham, New Hampshire 03824

Raytheon Company
Portmouth, R.I. 02871

UNIVERSITY of NEW HAMPSHIRE
DURHAM, NEW HAMPSHIRE. 03824

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Approved:

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Musa Yildiz - Technical Director

COOPERATING INSTITUTIONS

University of New Hampshire
Durham, New Hampshire 03824

Submarine Signal Division
Raytheon Company
Portsmouth, R.I. 02871

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LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

$\vec{A}$	Vector potential
$a_{\alpha,\beta}$	Function related to vertical wavenumber
a_{ij}	Rotation tensor
Arg{ }	Denotes argument of a complex quantity
A,B,C,D	functions in ζ-domain
c	Speed of propagation
d	Total differential operator
E_{ijmn}	Fourth-order tensor
$\hat{e}$	Unit vector
$\vec{f}$	Body force vector
G	Green's function in frequency domain
g	Green's function in time domain
$H(\omega)$	Spectrum of pulse
h	Thickness of a layer
I	Integral
i	Imaginary unit ($i^2 = -1$)
Im{ }	Denotes imaginary part of complex quantity
J_0	Bessel function of zeroth order
K	Bulk modulus of elasticity
k	Wavenumber
M	Matrix, also mass of a body
m	Density ratio ($m = \rho_1/\rho_0$)
m_{ij}	Element of M
P,Q,R,S	Functions of ζ

p	Pressure
$\vec{q}$	Heat conduction vector
R	Distance between source and receiver
r	Radial coordinate (also heat source)
$Re\{\ \}$	Real part of a quantity
S	Surface, usually of a volume element
s	Integration variable along branch cut or entropy per unit mass
T	Temperature
t	Time coordinate
U	Internal energy
$\vec{u}$	Displacement vector
V	Volume of a given element
$\vec{v}$	Velocity vector
W	Mechanical work
x	Cartesian coordinate, or $Re\{z\}$
y	Cartesian coordinate, or $Im\{z\}$
z	Cartesian coordinate, or complex variable ($z = x + iy$)
α	Longitudinal wave speed
β	Transverse wave speed
Γ	Denotes path of integration
γ	Ratio of two length scales
δ	Dirac delta function
δ_{ij}	Kronecker delta
ε	Pertubation parameter
ε_{ij}	Strain tensor

ζ	Variable in Fourier-Bessel transform domain and bulk viscosity
θ	Argument of complex variable $z (z = Re^{i\theta})$, local entropy production or angle of incidence
κ	Coefficient of heat conduction
λ	Lamé parameter, or wavelength
μ	Lamé parameter (Rigidity modulus)
Π	Denotes repeated multiplication
ρ	Mass density
σ_{ij}	Stress tensor
ϕ	Scalar potential
ω	Frequency
∂	Partial differential operator
∇	Gradient operator
$\cdot$	Denotes scalar multiplication of two vectors
$\times$	Denotes cross product of two vectors

Subscripts

α	Refers to longitudinal wave
β	Refers to transverse wave
i,j,k, l,m,n	Indices
$<$	Denotes minimum
$>$	Denotes maximum
0,1,2, ...j,...	Refers to layer

ABSTRACT

SOUND PROPAGATION IN A LIQUID
OVERLYING A VISCOELASTIC HALFSPACE

by

ALLEN H. MAGNUSON

The problem of acoustic subbottom sediment identification and classification is treated using an analytical approach. The purpose of the study is to develop expressions for the acoustic response in a liquid overlying a layered viscoelastic halfspace. After a review of related experimental results and earlier analytical studies, the fundamental governing laws are discussed and a linearization is applied. Linearized constitutive relations are developed for an elastic solid with superimposed damping (Voigt viscoelastic model). Vector displacement field equations are also derived for the inviscid fluid.

The vector field equations are simplified by separating the field into longitudinal and transverse parts and introducing scalar potential functions for the resulting polarizations. The response in the liquid is expressed as a Green's function due to the monopole point-source field excitation. Then the multi-layer problem is solved using the Green's function formalism, integral transforms and by matching boundary conditions at each interface between layers. A recurrence relation is developed for the potentials in adjoining viscoelastic layers. This recurrence relation is applied successively to eliminate the potentials

between the first and last viscoelastic layers. The result is suitable for computer studies due to the application of the recurrence relation.

Special cases of the multi-layer problem are developed and shown to be consistent with earlier results. The one viscoelastic layer (halfspace) case is analyzed in detail for both finite and infinite depth of the overlying liquid. The integral form is evaluated using complex variable techniques and high-frequency far-field approximations. The results are expressed as the sum of residue terms and branch line integrals. The branch line integrals are expressed as asymptotic series and the leading terms are evaluated for the infinite liquid depth. The result is shown to be applicable to the near-bottom case. In addition, a steepest descent integration is applied for moderate angles of incidence. The resulting response is the sum of the direct and reflected wave and a refracted wave that occurs for angles of incidence beyond a critical angle.

Application of the results to the subbottom identification problem is discussed. It is shown that the compressional and transverse wave propagation in the subbottom can be inferred from the results for the near-bottom infinite depth case. Additional information on the subbottom may be obtained from the moderate (oblique) incidence model.

TABLE OF CONTENTS

Page

I. INTRODUCTION

The extraction of mineral sediments from the continental shelf is becoming more feasible economically due to the steady depletion of resources on land. Interest in this area has created a demand for an inexpensive, rapid means for classifying underwater sediments. The purpose of the present work has been to develop understanding of the fundamental physical processes occurring in the ocean-subbottom system. The dynamical behavior of the system shown schematically in Figure 1 in response to acoustical input signals is of primary interest. The rationale of the present investigations is based on the premise that further advances in the remote classifications of marine sediments are dependent upon the development of more sophisticated and realistic analytical models. A model consisting of a coupled acoustic (ocean) and dynamic viscoelastic field (subbottom) is developed. Classification of sediments can then be accomplished in terms of the viscoelastic parameters of the subbottom. Enough progress has been made using this type of approach to justify further development.

A. Discussion of Results of Previous Investigators

Considerable experimental work has been done to classify or identify marine sediments. The most conclusive work to date is that of Breslau and Hamilton. Breslau [1] developed a relationship between subbottom reflectivity and the sediment porosity. Hamilton's results of extensive work done on the continental terrace and in the deep ocean are summarized in Reference [11]. He measured or computed the elastic properties of several types of marine sediments. His results for the continental

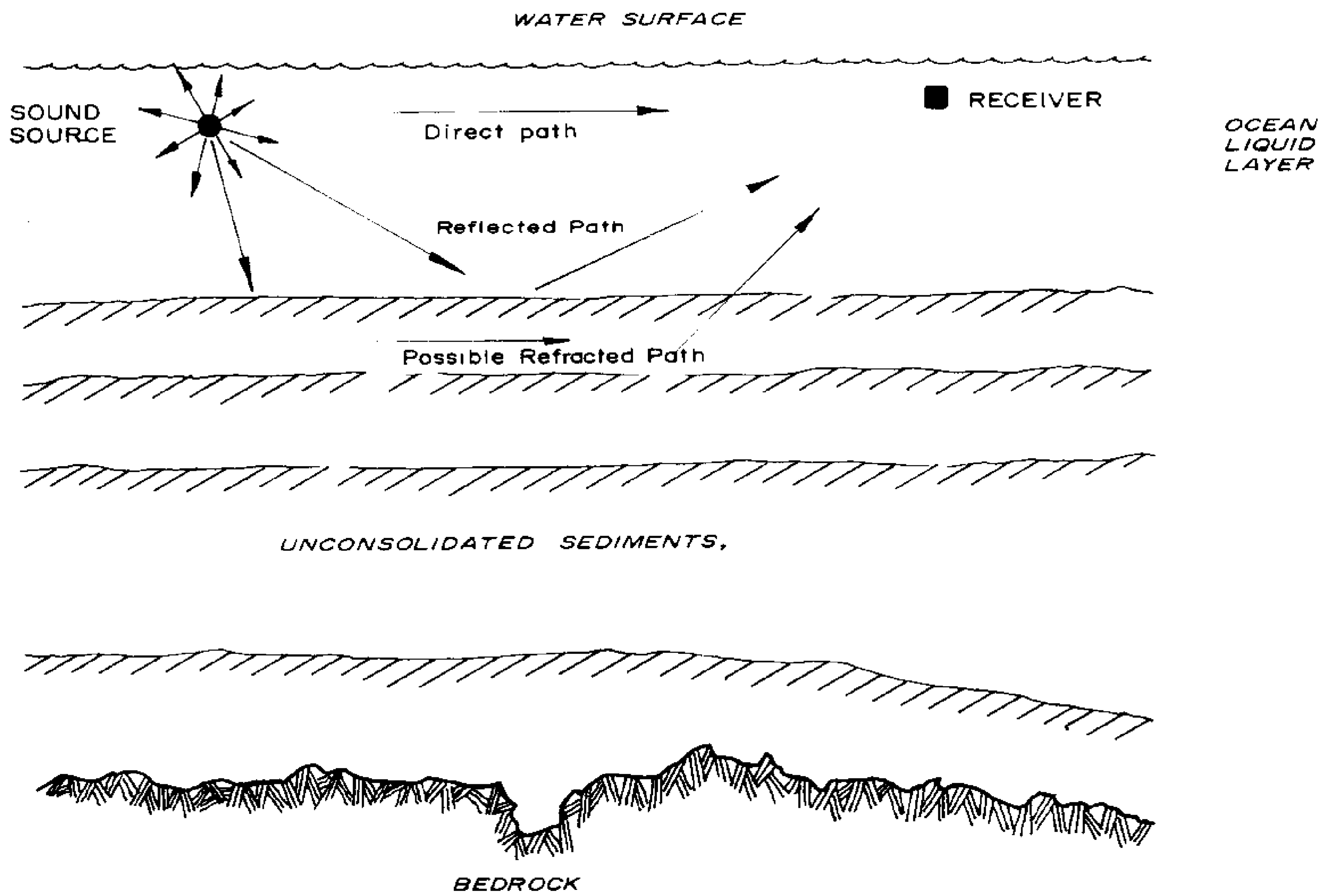


FIGURE 1

Diagram of System

terrace (shelf and slope) are presented in Table 1. Several general observations can be made from Hamilton's results. First, one sees that the compressional wave speed with the subbottom is slightly higher than the speed of sound in the overlying water. Secondly, the shear wave speed in the sediment is considerably slower than the water's speed of sound. In addition, the sediment density varies from about 1.5 to 2.0 times the water density.

Numerous investigations have been performed to model and measure the dissipative or damping properties of marine sediments. Krizek and Franklin [21] measured the energy dissipation in a soft clay. They found the energy dissipation to be independent of frequency from 0.1 to 30 Hz. Mizikos [30], [31] found that, for marine sands, the sliding contact between grains induced an amplitude and frequency independent phase lag between stress and strain. Other investigations have been conducted: for example, we cite the work of Hampton [12] and Wood and Weston [57]. Both obtained empirical relations for the attenuation or energy dissipation of sound in marine sediments. In any case, the damping mechanisms in marine sediments are not well understood and no universally accepted model for the damping exists valid over the frequency range of interest. The experiments generally conclude that marine sediments may be modeled as an elastic solid with small superimposed damping.

The experimental work on marine sediments has relied on relatively simple models to interpret the data. Breslau [1] used a plane-wave reflection coefficient model in conjuction with ray theory. In Hamilton's work [11] the compressional wave speed in the sediment was measured directly using probes. The shear wave velocities were computed

Table 1 Elastodynamic Wave Propagation Parameters for Marine Sediments on the Continental Terrace (From Hamilton [11])

Sediment Type	Mass Density $\rho (gm/cm^3)$	Longitudinal Wave Speed c_L (m/sec)	Transverse Wave Speed c_T (m/sec)
Sand			
Coarse	2.03	1836	250
Fine	1.98	1742	382
Very Fine	1.91	1711	503
Silty Sand	1.83	1677	457
Sandy Silt	1.56	1552	379
Sand-silt-clay	1.58	1578	409
Clayey silt	1.43	1535	364
Silty clay	1.42	1519	287

Note: Sound velocity in water $c_0 \cong 1501$ m/sec and water density $\rho_0 \cong 1.025$ g/cm^3 @ 20 m depth from Stoneley wave speed measurements using the close relation between the two waves [2].

The models used do not take into account several important effects: the first is the existence of sublayers in the sediment (shown in Figure 1) and the second is damping. Other effects such as nonuniformity of the sediment interfaces and sediment inhomogenieties are obviously not taken into account. The present work develops a more general acoustical-elastodynamic model with damping that takes into account the multiple layering. In addition, the three-dimensional nature of the problem is explicitly taken into account by modeling the acoustic stimulus as a point source in the liquid layer.

The analytical work developed here is an extension of seismological and geophysical investigations on the analysis of earth tremors in the deep ocean and response to underwater explosions. Ewing, Jardetsky and Press [5] present in a comprehensive survey the principal results of analytical studies in thisarea up to about 1957. The principal differences between seismological-geophysical modeling and acoustical subbottom identification arise due to differences in the time dependence of the stimuli to the system and in the ranges over which signals are monitored. Geophysical work usually uses an impulsive time dependence representing an explosion or a natural disturbance. In addition, distances from source to receiver are usually many times the water depth in the deep ocean. On the other hand, acoustic sounding of the subbottom in shallow water is usually done with a frequency-modulated pulse of short duration. The modulating frequencies are generally in the mid-audio range: i.e. from 1 KHz to 10 KHz. The ranges are usually much shorter, as both source and receiver are usually hung over the side of a single survey vessel. In addition, elastic wave propagation speeds in the deep ocean bottom differ qualitatively from those in sediments in shallow water (see Table 2). We see from the table that both the compressional and the shear wave in the rock bottom are higher than the liquid layer's sound velocity.

The relative differences in the physical parameters alter the viewpoints of the two activities. Generally in geophysical work modal behavior is predominant due to the large distances between stimulus and receiver. In acoustic sounding, reflected waves (generally the first return), refracted waves and interface waves are the significant effects picked up by sensors. The frequency content of seismological stimuli are usually at the low end of the spectrum, while the spectrum of a modulated acoustic signal is centered around the modulation frequency. High-frequency approximations may then be made in acoustical work. This facilitates evaluation of integral expressions for the response using asymptotic methods.

The earliest analytical work in the geophysical area dates back to Rayleigh [41] and Lamb [22]. The early work was extended and generalized by, among others, Nakano [36] and Lapwood [27]. Jeffries [17] first applied to the geophysical field the complex variable techniques developed by Sommerfeld [43] for electromagnetic wave propagation problems. Most of the later analytical work (including this) has been based on Sommerfeld's approach.

Pekeris [38] and Press and Ewing [40] investigated the response due to a point source in a liquid layer overlying a fluid and solid half-space respectively. Both neglected the effect of branch-line integrals,

Table 2 Elastodynamic Wave Propagation Parameters for the Ocean Floor (Rock) (From Ewing, et.al. [5], p. 162)

Type of Bottom	ρ_1/ρ_0	c_L/c_T	c_T/c_0	c_L/c_0
Granite	2.5	$\sqrt{3}$	2	$2\sqrt{3}$
Basaltic	3.0	$\sqrt{3}$	3	$3\sqrt{3}$

being primarily interested in the modal or waveguide-like part of the response. Honda and Nakamura [15] evaluated the branch line integrals for the problem treated by Press and Ewing. These integrals correspond to reflected and refracted waves.

The transmission of elastic waves through multi-layered media has been discussed for the plane-wave case by Thomson [53] and Haskell [13]. Thomson developed a matrix method that could be applied to an arbitrary number of layers using a recurrence relation. Haskell later removed an unnecessary restriction appearing in Thomson's formalism and computed group velocities for several assumed models of the earth's crust. Jardetzky [16] developed expressions for the period equation and the response due to a point source in an n-layered elastic halfspace. (Jardetzky's treatment also appears in Reference [6]).

B. Statement of the Problem

The purpose of this investigation is to develop expressions for the point-source acoustic response in a liquid layer overlying a layered solid halfspace. The results are to be developed systematically from fundamental principles. To keep the discussion as general as possible, expressions are developed in a frequency domain. The Fourier synthesis for specific input pulse shapes in the time domain is relatively straightforward and does not introduce any new fundamental insight into the problem.

The solid halfspace is assumed to consist of an arbitrary number of parallel horizontal layers. Each layer is assumed to be a linear homogeneous isotropic elastic solid with superimposed damping; e.g., a Voigt viscoelastic model. The frequency-domain response is to be calculated for values of physical parameters corresponding to typical marine sediments (see Table 1). The response will be interpreted physically and compared with the results of earlier investigators.

C. Method of Approach

The overall approach to the problem is analytical. Approximations and simplifications based on physical arguments are made to facilitate discussion in cases where analytical complexity precludes a general treatment. An effort has been made to develop results in the most general form, after which simplifications and special cases are discussed.

In Chapter II we derive the dynamic equations for viscoelastic solid. The equations are developed from fundamental conservation laws and thermodynamical considerations. Suitable constitutive relations are developed. Linearization is applied based on small disturbances from a uniform equilibrium state.

Chapter III is devoted to the simplification of the vector field equations derived in the preceding chapter. The vector field equations are broken down into longitudinal and transverse parts and the Fourier transform in time is applied. Solutions to the vector field equations are developed using scalar potential functions. Finally the stress and displacement fields are expressed in terms of the scalar functions. The field excitation in the liquid layer is developed using a point source model.

In Chapter IV the boundary value problem for the general n-layered solid halfspace with an overlying liquid layer is solved using a matrix formalism combined with a recurrence relation. Integral transform methods are applied, enabling the boundary conditions to be evaluated as algebraic expressions in the transform domain. The response is expressed as a Green's function in integral form.

Special cases of the Green's function are evaluated in Chapters V and VI. The homogeneous solid halfspace with overlying liquid layer is treated in Chapter V. Chapter VI is concerned with the liquid halfspace over a homogeneous solid halfspace. In both chapters approximate expressions for the frequency domain Green's functions are obtained using asymptotic techniques.

Finally, in Chapter VII the results are summarized and compared with previous results. New results are discussed and interpreted physically where possible.

II. THE DYNAMIC FIELD EQUATIONS

The dynamic field equations are derived for waves propagating in elastic and damped elastic media. The usual conservation laws are written and linearized. Suitable constitutive relations are developed for small disturbances from an equilibrium state. Finally, the dynamic field equations are developed by combining the linearized conservation laws with the constitutive relations.

Much of the discussion and development in this chapter uses the Cartesian tensor notation for convenience. Later, a curvilinear coordinate system is introduced due to symmetries existing in the field. The Cartesian tensor notation is used in the development of the field equations due to the relative ease of computation using this representation. Final results are later converted to an invariant notation for use in the orthogonal curvilinear coordinate system.

A. Deformation Analysis

We introduce a displacement vector for a continuum as $\vec{u} = \vec{u}(\vec{x})$. That is, the displacement vector is a field quantity defined throughout the Euclidean 3-space $\vec{x}$. We define the displacement as the distance a material point in space moves from some original undeformed configuration

$$\vec{u}(\vec{x},t) = \vec{r}'[\vec{X}(\vec{x}),t] - \vec{X}(\vec{x}), \tag{2.1}$$

where $\vec{X}$ maps the particle into the Euclidean 3-space in the undeformed configuration, and $\vec{r}'$ is the position of the same particle after the field has undergone a deformation. We expand $\vec{r}'$ spatially as follows:

$$r_i'(\vec{x}) = X_i + \left.\frac{\partial r_i'}{\partial x_j}\right|_X dx_j + \ldots , \tag{2.2}$$

where the differentiation is performed for a fixed particle and

$$dx_j = x_j - X_j .$$

Substituting eq. (2.2) into (2.1) gives for the deformation

$$u_i(\vec{x},t) = \left.\frac{\partial r_i'}{\partial x_j}\right|_{\vec{X}} dx_j + \ . \ . \ . \tag{2.3}$$

For small displacements one may also write a chain rule as follows:

$$u_i = \left(\frac{\partial u_i}{\partial x_j}\right) dx_j + \ . \ . \ . \ , \tag{2.4}$$

where the quantity $(\partial u_i/\partial x_j)$ is the deformation gradient, a second-order tensor. We may write the deformation gradient as follows from eqs. (2.4) and (2.3)

$$\frac{\partial u_i}{\partial x_j} = \left.\frac{\partial r_i'}{\partial x_j}\right|_{\vec{X}} .$$

One may introduce the infinitesimal strain tensor ε_{ij} by taking the symmetric part of the deformation gradient

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \tag{2.5}$$

In addition, the velocity $\vec{v}$ may be introduced by taking the material time derivative of the displacement

$$\vec{v} = \frac{d}{dt}\vec{u} = \dot{\vec{u}} \ . \tag{2.6}$$

Here the material or total time derivative is interpreted in Cartesian notation as

$$\frac{d(\ )}{dt} = \frac{\partial(\ )}{\partial t} + \vec{v}\cdot\nabla(\ ) = \frac{\partial(\ )}{\partial t} + v_i \frac{\partial(\ )}{\partial x_i} \ . \tag{2.7}$$

A velocity gradient may be introduced as

$$\frac{\partial v_i}{\partial x_j} \ .$$

We denote the rate of strain tensor $\dot{\varepsilon}_{ij}$ as the symmetric part of the velocity gradient

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{2.8}$$

We note that for infinitesimal strains, the rate of strain tensor is identical to the time derivative of the strain tensor.

Later in the chapter, we take the elements of the strain tensor to be infinitesimal as the basis for the linearization of the governing equations. This implies for wave type propagation that the deformations are small. That is, in wave propagation, the disturbance is limited to a small region in space (the length of the wave pulse). Uniform disturbances like thermal expansion, which imply large large deformations, are ruled out. The displacement $\vec{u}$ may be taken as small relative to a characteristic length scale in the field, such as the length of the propagating pulse. The elements of the rate of strain tensor are also taken as small, which implies that the field velocity is small relative to some velocity scale such as a speed of sound propagation in the medium.

B. Conservation Laws

The medium or continuum of interest is governed by four conservation laws. We restrict the medium to be one in which only mechanical and thermodynamic effects are significant. In addition, we rule out the possibility of the medium's sustaining either a body couple or a couple-stress. In this case, the governing laws are the conservation of mass, linear momentum, angular momentum and energy. The conservation of mass may be written as

$$\frac{\partial \rho}{\partial t} + \partial_i(\rho v_i) = 0, \tag{2.9}$$

or in vector notation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0, \tag{2.9a}$$

where ρ is the mass density. The conservation of linear momentum implies that

$$\rho\frac{d}{dt} v_j - \partial_i\sigma_{ij} = \rho f_j , \tag{2.10}$$

where σ_{ij} is the stress tensor and f_j is the body force per unit mass. Equation (2.10) is also referred to as the equation of motion.

The conservation of angular momentum simply requires that the stress tensor be symmetric, or

$$\sigma_{ij} = \sigma_{ji} \tag{2.11}$$

in the absence of body couples and couple stresses. This implies that only six independent elements of the stress tensor exist.

The last conservation law may be written as

$$\rho\dot{U} - \rho r + \partial_i q_i - \sigma_{ij}\dot{\varepsilon}_{ij} = 0, \tag{2.12}$$

where U is the specific internal energy, r is the heat supply per unit mass and q_i is the heat flux (efflux) vector. Equation (2.12) is the energy balance statement.

In addition to the four conservation laws, a generalization of the second law of thermodynamics may be introduced. The second law governs the local growth of entropy for elements of mass moving with the medium. We use a treatment based on Sommerfeld [44], except that we consider the more general case where motion of the medium and mechanical work effects are taken into account.

One starts by writing the energy balance statement for a reversible process as follows:

$$dU = \delta q - \delta W, \tag{2.13}$$

where dU (the change in internal energy) is a perfect differential, δq is the change in heat energy and δW is the increment of work done. All terms are per unit mass, and the "system" is an element of mass moving with the medium. One introduces entropy by writing the heat added as

$$\delta q = T ds, \tag{2.14}$$

where T is the absolute temperature and s is the entropy per unit mass. One may write the mechanical work term due to internal stresses as

$$\delta W = \frac{-1}{\rho}(\sigma_{ij})_R \, d\varepsilon_{ij}, \tag{2.15}$$

where $(\sigma_{ij})_R$ is the component of stress associated with a reversible process. In general, the stress may be written

$$\sigma_{ij} = (\sigma_{ij})_R + \sigma_{ij}', \tag{2.16}$$

where σ_{ij}' is the component of stress due to dissipation or irreversible processes. Substituting eqs. (2.14) and (2.15) into (2.13) gives

$$\rho dU = \rho T ds + (\sigma_{ij})_R \, d\varepsilon_{ij}. \tag{2.17}$$

Introducing material time rate derivatives for the differentials in eq. (2.17) yields

$$\rho \dot{U} = \rho T \dot{s} + (\sigma_{ij})_R \, \dot{\varepsilon}_{ij} \tag{2.17a}$$

One obtains the rate of production of entropy by rearranging eq. (2.17a) as follows:

$$\rho \dot{s} = \frac{1}{T} \rho \dot{U} - \frac{1}{T} (\sigma_{ij})_R \, \dot{\varepsilon}_{ij}. \tag{2.18}$$

One applies the expression for the internal energy rate from eq. (2.12) and eq. (2.16) to the entropy production relation to obtain:

$$\rho \dot{s} = \frac{1}{T} \rho r - \frac{1}{T} \operatorname{div} \vec{q} + \frac{1}{T} \sigma_{ij}' \, \dot{\varepsilon}_{ij}. \tag{2.19}$$

The heat conduction term may be written as two terms, giving for eq. (2.19):

$$\rho\dot{s} + \mathrm{div}\,(\frac{\vec{q}}{T}) = \frac{\rho r}{T} - \frac{1}{T^2}(\vec{q}\cdot\,\mathrm{grad}\,T) + \frac{1}{T}\,\sigma'_{ij}\,\dot{\varepsilon}_{ij} \tag{2.19a}$$

To interpret eq. (2.19a), we consider a reversible process, where grad T = 0 and $\sigma'_{ij} = 0$, e.g. no temperature gradient or mechanical dissipation. In this case, eq. (2.19a) reduces to

$$\rho\dot{s} + \mathrm{div}\,(\frac{\vec{q}}{T}) = \frac{\rho r}{T} \tag{2.19b}$$

One may integrate eq. (2.19b) over a mass element with volume V and enclosing surface area A.

$$\frac{d}{dt}\int_V s\,dm + \int_A \frac{\vec{q}\cdot\hat{n}}{T}\,da = \int_V \frac{r\,dm}{T}, \tag{2.20}$$

where $dm = \rho dV$ and $\hat{n}$ is a unit outward normal vector from the element of surface area da. One notes that eq. (2.20) is a statement of the conservation of entropy. The first term on the left-hand side is the time rate of change of entropy in the mass element, the second is the efflux of entropy through the boundary A and the right-hand side is the entropy source term. We recall that for a reversible process the entropy is conserved. Returning to the local entropy production statement for the irreversible process [eq. (2.19a)], one sees that the last two terms on the right-hand side must represent local entropy production. We write θ, the local entropy production per unit volume as

$$\theta = -\frac{1}{T^2}\,(\vec{q}\cdot\,\mathrm{grad}\,T) + \frac{1}{T}\,\sigma'_{ij}\,\dot{\varepsilon}_{ij}. \tag{2.21}$$

The second law of thermodynamics requires that the local entropy production be positive for an irreversible process, or

$$\Theta > 0. \tag{2.22}$$

For a reversible process, one has

$$\theta = 0$$

from eq. (2.19b). Each term on the right-hand side of eq. (2.21) must be positive for an irreversible process, since for a process with no mechanical dissipation ($\dot{\varepsilon}_{ij} = 0$),

$$\Theta = \Theta_q = -\frac{1}{T^2}(\vec{q}\cdot \text{grad } T) > 0, \tag{2.22a}$$

and for a process with no heat flow ($\vec{q} = 0$), one has

$$\Theta = \Theta_m = \frac{1}{T}\sigma'_{ij}\,\dot{\varepsilon}_{ij} > 0. \tag{2.22b}$$

From eqs. (2.21) and (2.22), we have

$$\Theta_q + \Theta_m > 0$$

and

$$\Theta_q > 0$$

$$\Theta_m > 0.$$

If one applies Fourier's law of heat conduction, we may write for the heat flux vector

$$\vec{q} = -\kappa\,\text{grad } T, \tag{2.23}$$

where κ is the coefficient of heat conduction. The entropy production due to heat conduction becomes, from eqs. (2.22a) and (2.23):

$$\Theta_q = \frac{\kappa}{T^2}(\text{grad } T)\cdot(\text{grad } T) > 0. \tag{2.24}$$

From eq. (2.24) one sees that the coefficient of heat conduction must be positive.

One may obtain a relation for σ'_{ij} (the dissipative part of the stress) by examining the entropy production due to the mechanical dissipative process, e.g. eq. (2.22b). If one writes a linear relation between σ'_{ij} and the strain rate $\dot{\varepsilon}_{ij}$, one has

$$\sigma'_{ij} = E'_{ijmn}\,\dot{\varepsilon}_{mn} \tag{2.25}$$

Applying this to eq. (2.22b) gives

$$\theta_m = \frac{1}{T} E'_{ijmn} \dot{\varepsilon}_{mn} \dot{\varepsilon}_{ij} > 0, \tag{2.26}$$

a quadratic form in the strain rate tensor. For an isotropic medium, the fourth-order tensor E'_{ijmn} must also be isotropic (see Appendix A). One notes the symmetry

$$E'_{ijmn} = E'_{mnij}$$

from the quadratic form (2.26). We write for an isotropic medium

$$E'_{ijmn} = \lambda' \delta_{ij} \delta_{mn} + \mu'(\delta_{im}\delta_{jn} + \delta_{in} \delta_{jm}). \tag{2.27}$$

Applying eq. (2.27) to (2.25) gives

$$\sigma'_{ij} = \lambda' \dot{\varepsilon}_{\ell\ell} \delta_{ij} + 2\mu' \dot{\varepsilon}_{ij}. \tag{2.28}$$

It is convenient to introduce into eq. (2.28) a bulk viscosity ζ defined as follows:

$$\zeta = \frac{3\lambda' + 2\mu'}{3} \tag{2.29}$$

We write eq. (2.28) in terms of the bulk viscosity as

$$\sigma'_{ij} = \zeta\dot{\varepsilon}_{\ell\ell} \delta_{ij} + 2\mu'(\dot{\varepsilon}_{ij} - \frac{1}{3} \delta_{ij} \dot{\varepsilon}_{\ell\ell}), \tag{2.28a}$$

where the second term on the right-hand side is traceless. Now one may write the strain rate tensor as the sum of a non-deviatoric and a deviatoric (traceless) component:

$$\dot{\varepsilon}_{ij} = \frac{1}{3} \delta_{ij} \dot{\varepsilon}_{\ell\ell} + (\dot{\varepsilon}_{ij} - \frac{1}{3} \delta_{ij} \dot{\varepsilon}_{\ell\ell}). \tag{2.30}$$

Applying eqs. (2.28a) and (2.30) to eq. (2.22b) gives

$$\Theta_m = \frac{1}{T}[\zeta(\dot{\varepsilon}_{\ell\ell})^2 + 2\mu' (\dot{\varepsilon}_{ij} - \frac{1}{3} \delta_{ij} \dot{\varepsilon}_{\ell\ell})^2] > 0 \tag{2.31}$$

Eq. (2.31) requires that

$$\zeta = \frac{3\lambda' + 2\mu'}{3} > 0$$

and $\mu'>0$, since the mechanical entropy production is the sum of two independent quadratic terms: one associated with dilatational motion (change in volume) and the other with shearing-type motion.

One sees that the second law of thermodynamics as stated in eqs. (2.16, 21 and 22) requires that the heat conduction coefficient κ and the viscosities ζ and μ' be positive. In addition, the second law yields directly the constitutive relation for the dissipative part of the stress tensor [eqs. (2.28) or (2.28a)].

C. Linearization of Governing Equations

The type of disturbance to the medium we wish to analyze is a wave or series of waves. We assume that this disturbance is relatively weak so that nonlinear effects are negligible. The disturbance may be considered to be limited in extent spatially. That is, the disturbance is a wave front due to some initial concentrated impulse. We consider the medium to be at rest and in an undeformed state in the absence of the disturbance, where $\vec{u} = 0$ and $\vec{v} = 0$. The undisturbed temperature and density may be denoted as T_0 and ρ_0, respectively. In addition, the undisturbed field is assumed to be uniform spatially, so that, if we denote the disturbance effects by a prime, we may write

$$\vec{u} = \vec{u}'(\vec{x}, t) \tag{2.32}$$

$$\vec{v} = v'(\vec{x}, t)$$

$$\rho = \rho_0 + \rho'(\vec{x}, t)$$

$$T = T_0 + T'(\vec{x}, t)$$

In eq. (2.32) we assumed the primed quantities are small in the following sense:

$$\rho' << \rho_0$$

$$T' << T_0$$

$$v' << c$$

$$u' << c\nabla t,$$

where c is a speed of propagation in the medium, and ∇t is a time scale such as the wave pulse duration. The strain tensor elements and the rate of strain tensor elements are also taken as small, or

$$|\varepsilon_{ij}| \ll 1$$

$$|\dot{\varepsilon}_{ij}| \ll \frac{1}{\nabla t} .$$

We linearize the governing equations (2.9) and (2.10) by retaining only first-order terms in the primed quantities listed in eq. (2.32). Equation (2.9) reduces to

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{v} = 0. \tag{2.33}$$

The equation of motion [eq. (2.10)] reduces to

$$\rho_0 \frac{\partial v_j'}{\partial t} - \partial_i \sigma_{ij} = \rho_0 f_j, \tag{2.34}$$

where the material time derivative reduces to a local time derivative due to the linearization, or

$$\frac{\partial(\ )}{\partial t} = \frac{d(\ )}{dt}.$$

The velocity may be written as $\vec{v} = \frac{\partial \vec{u}}{\partial t}$ for the linearized case, converting eq. (2.18) to

$$\rho_0 \frac{\partial^2 u_j}{\partial t^2} - \partial_i \sigma_{ij} = \rho_0 f_j. \tag{2.34a}$$

The energy conservation statement [eq. (2.12)] and the entropy production equation (2.19) may be linearized by replacing the density ρ by ρ_0 and by interpreting the time derivatives as local derivatives.

D. Constitutive Equations

We wish to relate the stress tensor σ_{ij} to the independent variables in the thermo-mechanical field. One may consider the displacement and its time and spatial derivatives and the temperature and its derivatives as the independent variables. We write the stress in functional form as follows:

$$\sigma_{ij} = \tilde{\sigma}_{ij}\ (u_k,\ v_k,\ \partial_\ell u_k,\ \partial_\ell v_k,\ T,\ \partial_k T,\ +\ .\ .\ .) \tag{2.35}$$

One may eliminate the displacement, velocity and the skew-symmetric parts of the displacement and velocity gradients from the functional form by ruling out dependence of the stress on the rigid-body motion of the medium. This simplifies eq. (2.35) to the following:

$$\sigma_{ij} = \tilde{\sigma}_{ij}\ (\varepsilon_{lk},\ \dot{\varepsilon}_{lk},\ T,\ \partial_k T). \tag{2.35a}$$

The linearization [eq. (2.32)] implies that the stress is only a function of the ambient temperature T_0, reducing eq. (2.35a) to

$$\sigma_{ij} = \tilde{\sigma}_{ij}\ (\varepsilon_{lk},\ \dot{\varepsilon}_{lk},\ T_o), \tag{2.35b}$$

where the time derivative reduces to a local derivative for the strain rate.

We recall from the thermodynamic discussion that stress was broken into a reversible part and a dissipative part [eq. (2.16)]:

$$\sigma_{ij} = (\sigma_{ij})_R + \sigma'_{ij}.$$

A constitutive relation has already been obtained as a consequence of the second law of thermodynamics for the dissipative part of the stress [eqs. (2.28 and 28a)]. All that remains is to obtain a constitutive relation for the reversible component $(\sigma_{ij})_R$. We refer to the energy balance statement for a reversible process

$$dU = \delta q - \delta W. \tag{2.13}$$

Setting the heat increment to zero and introducing eq. (2.15) to eq.(2.13) gives

$$dU = -dW = \frac{1}{\rho}\ (\sigma_{ij})_R\ d\ \varepsilon_{ij} \tag{2.36}$$

Linearizing this by writing $\rho \cong \rho_o$ gives

$$\rho_o dU = +\ (\sigma_{ij})_R\ d\varepsilon_{ij} \tag{2.37}$$

One sees that the left-hand side of eq. (2.37) is the strain or deformation energy stored per unit volume, and dU must be a perfect differential for a reversible process. If the right-hand side of eq. (2.37) is to be a perfect differential, one may write the stress as a linear function of the strain

$$(\sigma_{ij})_R = E_{ijmn}\, E_{mn}. \tag{2.38}$$

Applying this relation to eq. (2.37) gives

$$\rho_o dU = E_{ijmn}\, \varepsilon_{mn}\, d\varepsilon_{ij} \tag{2.39}$$

Formally integrating eq. (2.39) gives

$$\rho_o U = \frac{1}{2} E_{ijmn}\, \varepsilon_{mn}\, \varepsilon_{ij}, \tag{2.40}$$

a quadratic in the strain tensor. The integration constant U_o is set to zero for zero strain. The quadratic form of eq. (2.40) implies the following symmetry:

$$E_{ijmn} = E_{mnij}.$$

We are interested in an isotropic medium, so we may write in a manner analogous to eq. (2.27) the following

$$E_{ijmn} = \lambda\delta_{ij}\,\delta_{mn} + \mu(\delta_{im}\,\delta_{jn} + \delta_{in}\,\delta_{jm}), \tag{2.41}$$

where λ and μ are Lamé constants. We substitute this result into eq. (2.38), giving

$$(\sigma_{ij})_R = \lambda\varepsilon_{\ell\ell}\,\delta_{ij} + 2\mu\,\varepsilon_{ij} \tag{2.42}$$

For the isotropic medium, eq. (2.40) for the internal energy due to deformation becomes:

$$\rho_o U = \frac{1}{2}\,[\lambda(\varepsilon_{\ell\ell})^2 + 2\mu(\varepsilon_{ij})^2] \tag{2.43}$$

One notes from eq. (2.37) that the reversible part of the stress is related to a thermodynamic derivative as follows:

$$(\sigma_{ij})_R = \rho_o \frac{dU}{d\varepsilon_{ij}}, \tag{2.44}$$

where the derivative is taken for constant entropy, since we restricted $\delta q = Tds$ to be zero. The Lamé constants λ and μ in eq. (2.42) are then taken for adiabatic deformations (e.g., no heat conduction).

We summarize by writing the stress tensor in its most general form from eqs. (2.16), (2.28) and (2.42) as follows:

$$\begin{aligned}\sigma_{ij} &= (\sigma_{ij})_R + (\sigma'_{ij}) = \\ &= (\lambda\varepsilon_{\ell\ell}\delta_{ij} + 2\mu\varepsilon_{ij}) + (\lambda'\dot{\varepsilon}_{\ell\ell}\delta_{ij} + 2\mu'\dot{\varepsilon}_{ij}) \\ &= (\lambda + \lambda'\frac{\partial}{\partial t})\varepsilon_{\ell\ell}\delta_{ij} + 2(\mu + \mu\,\frac{\partial}{\partial t})\varepsilon_{ij}\end{aligned} \tag{2.45}$$

Equation (2.45) is the constitutive relation for a damped (viscoelastic) isotropic linear solid undergoing small deformations. The constitutive relation for the elastic solid may be obtained from eq. (2.45) by setting the dissipative terms to zero, or

$$\lambda' = 0.$$

and

$$\mu' = 0.$$

Similarly, the constitutive relation for the undamped liquid may be obtained by setting both the rigidity μ to zero in eq. (2.45) and the damping terms, yielding

$$\sigma_{ij} = (\sigma_{ij})_R = \lambda\varepsilon_{\ell\ell}\delta_{ij} \tag{2.46}$$

Recalling the usual relation between the pressure fluctuation for an inviscid liquid and the stress:

$$\sigma_{ij} = -\,p'\delta_{ij}, \tag{2.47}$$

one sees that

$$p' = -\,\lambda\varepsilon_{\ell\ell}.$$

That is, the pressure fluctuation is proportional to the adiabatic volume change or dilatation.

One may obtain the linearized equations of motion for the viscoelastic solid, the elastic solid or the inviscid fluid by substituting the appropriate constitutive relation into the momentum conservation equation (2.34a). For the viscoelastic solid, we use the constitutive equation (2.45), giving

$$\rho_o \partial_t^2 u_i - [(\lambda + \mu)+(\lambda' + \mu')\partial_t]\partial_i \partial_\ell u_\ell - [\mu + \mu' \partial_t]\partial_j \partial_j u_i = \rho_o f_i \qquad (2.48)$$

or, in vector notation

$$\rho_o \partial_t^2 \vec{u} - [(\lambda + \mu)+(\lambda' + \mu')\partial_t]\nabla(\nabla\cdot\vec{u}) - (\mu + \mu'\partial_t)\nabla^2\vec{u} = \rho_o \vec{f}. \qquad (2.48a)$$

Now ∇^2 operating upon a vector is not an invariant vector form [44]. Instead, it must be interpreted as an operator of the form:

$$\nabla^2(\ ) = \nabla\nabla\cdot(\ ) - \nabla x(\nabla x(\ )). \qquad (2.49)$$

Using this identity allows us to express eq. (2.48) in an invariant form:

$$\rho_0 \partial_t^2 \vec{u} - [(\lambda+2\mu) + (\lambda' + 2\mu')\partial_t]\nabla(\nabla\cdot\vec{u}) + [\mu+\mu'\partial_t]\nabla x(\nabla x\vec{u}) = \rho_0 \vec{f}\ . \qquad (2.50)$$

The equation of motion for the linear isotropic elastic solid is obtained as a special case of eq. (2.50) by setting the damping terms to zero, or

$$\lambda' = 0$$

and

$$\mu' = 0.$$

Applying the constitutive relation for the inviscid liquid [eq. (2.47)] to the linearized equation of motion (2.34a) gives:

$$\rho_o \frac{\partial^2 u}{\partial t^2} + \nabla p' = \rho_o \vec{f}. \qquad (2.51)$$

One may manipulate eq. (2.51) into the acoustic wave equation as follows. The pressure p is written as a function of the density and entropy (e.g., the thermo-mechanical equation of state):

$$p = p\ (\rho,s). \qquad (2.52)$$

We expand eq. (2.52) about the ambient state p_o, ρ_o, s_o as follows:

$$p = p_o + (\frac{\partial p}{\partial s})(s-s_o) + (\frac{\partial p}{\partial \rho})(\rho-\rho_o) + \ldots \tag{2.52a}$$

From this expansion and eq. (2.32) one may express the pressure fluctuation as

$$p' = (\frac{\partial p}{\partial s})_\rho s' + (\frac{\partial p}{\partial \rho})_s \rho' + \ldots \tag{2.53}$$

For an adiabatic (and isentropic) process, we write

$$p' = (\frac{\partial p}{\partial \rho})_s \rho' \tag{2.53a}$$

The adiabatic sound velocity c_o is defined in terms of the thermodynamic derivative as follows:

$$c_o{}^2 = (\frac{\partial p}{\partial \rho})_s . \tag{2.54}$$

Applying eqs. (2.54) and (2.53a) to eq. (2.51) gives

$$\rho_0 \partial_t^2 \vec{u} + c_0^2 \nabla \rho' = \rho_0 \vec{f}. \tag{2.55}$$

Equation (2.55) may be written in a wave operator form after applying the linearized equation of continuity

$$\partial_t \rho' + \rho_0 \nabla \cdot (\partial_t \vec{u}) = 0. \tag{2.33}$$

Taking the divergence of eq. (2.55) gives

$$\rho_0 \partial_t^2 (\nabla \cdot u) + c_0^2 \nabla^2 \rho' = \rho_0 \nabla \cdot \vec{f} . \tag{2.56}$$

The divergence of $\vec{u}$ is, from eq. (2.33)

$$\nabla \cdot \vec{u} = - \frac{1}{\rho_0} \rho' .$$

We eliminate $\nabla \cdot \vec{u}$ from eq. (2.56), giving

$$(\nabla^2 \rho' - \frac{1}{c_0^2} \frac{\partial^2 \rho'}{\partial t^2}) = \frac{\rho_0 \nabla \cdot \vec{f}}{c_0^2} . \tag{2.57}$$

Equation (2.57) is a scalar inhomogeneous wave equation in ρ', the density fluctuation. This is the classical result of theoretical acoustics [24], [34]. We see here how the wave operator is developed from a thermodynamic state equation for the pressure and from the linearization process.

III. SIMPLIFICATION OF THE VECTOR FIELD EQUATIONS

In the previous chapter, linearized vector field equations were developed for the solid medium (2.50) and for the inviscid fluid (2.51). The field equations are simplified in this chapter using techniques developed by Hansen (a discussion of these appears in Morse and Feshbach [32]) for electromagnetic wave propagation problems. The application of these techniques to wave propagation in linearized solids is due to A. Yildiz [58]. After simplifying the field equations for the solid to scalar Helmholtz equations, a Green's function formalism is introduced to model the acoustic field (the linearized inviscid fluid) due to the monopole type of excitation.

A. The Elastic Solid

We write the field equation for the elastic solid from eq. (2.50) as

$$\rho_0 \partial_t^2 u + \mu[\nabla \times (\nabla \times \vec{u})] - (\lambda + 2\mu)\nabla(\nabla \cdot u) = \rho_0 \vec{f} \tag{3.1}$$

This is in an invariant form, so it applies to any orthogonal curvilinear or Cartesian coordinate system. Taking the divergence of eq. (3.1) gives

$$[\rho_0 \partial_t^2 - (\lambda + 2\mu)\nabla^2]\nabla \cdot \vec{u} = \rho_0 \nabla \cdot \vec{f}, \tag{3.1a}$$

where the second term in eq. (3.1) drops out because $\nabla \cdot \nabla \times (\ ) = 0$. We manipulate eq. (3.1a) into the form

$$[\nabla^2 - \frac{1}{c_L^2} \partial_t^2] \nabla \cdot \vec{u} = - \frac{1}{c_L^2} (\nabla \cdot f), \tag{3.1b}$$

where

$c_L^2 = \frac{\lambda+2\mu}{\rho_0}$, and c_L is the longitudinal sound velocity.

We note that since the divergence of a vector is a scalar, eq. (3.1b) is a scalar inhomogeneous wave equation.

Taking the curl of eq. (3.1) gives

$$[\rho_0\partial_t^2 + \mu\nabla x\nabla x](\nabla x\vec{u}) = \rho_0\nabla x\vec{f}, \tag{3.2}$$

where we have used the result curl grad ( ) = 0. One writes eq. (3.2) in the following form:

$$[\nabla x\nabla x + \frac{1}{c_T^2}\partial_t^2](\nabla x\vec{u}) = \frac{1}{c_T^2}(\nabla x\vec{f}), \tag{3.2a}$$

where $c_T^2 = \mu/\rho_0$, and c_T is the transverse velocity.
Recalling eq. (2.49) one notes that eq. (3.2a) can be written as a vector wave equation

$$[\nabla^2 - \frac{1}{c_T^2}\partial_t^2](\nabla x\vec{u}) = -\frac{1}{c_T^2}(\nabla x\vec{f}) \tag{3.2b}$$

It is convenient at this point to decompose the displacement vector and the body force vector into longitudinal and transverse parts as follows:

$$\begin{aligned} \vec{u} &= \vec{u}_L + \vec{u}_T \\ \vec{f} &= \vec{f}_L + \vec{f}_T, \end{aligned} \tag{3.3}$$

where the subscript L refers to the longitudinal component and T to the transverse. We set

$$\begin{aligned} &\nabla\cdot\vec{u}_T = 0, \quad \nabla\cdot\vec{f}_T = 0 \\ &\nabla x\vec{u}_L = 0, \quad \nabla x\vec{f}_L = 0 \end{aligned} \tag{3.4}$$

The longitudinal field is then defined as the curl-less component and the transverse as the divergence-less part. Equations (3.1b) and (3.2b) may be written

$$[\nabla^2 - \frac{1}{c_L^2}\partial_t^2](\nabla\cdot\vec{u}_L) = -\frac{1}{c_L^2}(\nabla\cdot\vec{f}_L) \tag{3.5}$$

$$[\nabla x\nabla x + \frac{1}{c_T^2}\partial_t^2](\nabla x\vec{u}_T) = +\frac{1}{c_T^2}(\nabla x\vec{f}_T) \tag{3.6}$$

Eqs. (3.5) and (3.6) show why c_L was termed the longitudinal and c_T the transverse sound propagation speed. The longitudinal component of the field propagates at speed c_L and the transverse at c_T. We recall that taking the divergence of the field equation (3.1) annihilated the transverse component of the field and, similarly, taking the curl eliminated the longitudinal. Taking the divergence and curl of a vector field separates the field into longitudinal and transverse polarizations.

We manipulate eqs. (3.5) and (3.6) as follows:

$$\nabla\cdot[(\nabla^2 - \frac{1}{c_L^2}\partial_t^2)\vec{u}_L + \frac{1}{c_L^2}\vec{f}_L] = 0 \tag{3.5a}$$

$$\nabla x[\nabla x(\nabla x\vec{u}_T) + \frac{1}{c_T^2}\partial_t^2\vec{u}_T - \frac{1}{c_T^2}\vec{f}_T] = 0 \tag{3.6a}$$

We may also write, from eq. (3.4),:

$$\nabla x[(\nabla^2 - \frac{1}{c_L^2}\partial_t^2)\vec{u}_L + \frac{1}{c_L^2}\vec{f}_L] = 0 \tag{3.5b}$$

$$\nabla\cdot[\nabla x(\nabla x\vec{u}_T) + \frac{1}{c_T^2}\partial_t^2\vec{u}_T - \frac{1}{c_T^2}\vec{f}_T] = 0 \tag{3.6b}$$

From vector analysis, we know that if both the divergence and curl of a vector field vanishes, then the field itself must vanish. The quantities inside the brackets in eqs. (3.5a and b) and (3.6a and b) must vanish, leaving

$$[\nabla^2 - \frac{1}{c_L^2}\partial_t^2]\vec{u}_L = -\frac{1}{c_L^2}\vec{f}_L \tag{3.7}$$

and

$$[\nabla x \nabla x + \frac{1}{c_T^2}\partial_t^2]\vec{u}_T = \frac{1}{c_T^2}\vec{f}_T \ . \tag{3.8}$$

Both of these are now vector wave equations, and we note that the ∇^2 operator in eq. (3.7) must be interpreted as $\nabla^2 = \nabla\nabla\cdot(\ )$ from eq. (2.49), since it operates on a longitudinal field.

Introduction of Fourier Transform in Time

Equations (3.7 and 8) are differential forms in space and time. Solutions are obtained more easily by transforming in time first. This reduces the differential form to an algebraic form in the transform (frequency) domain. One introduces the following Fourier transform pair:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \tag{3.9a}$$

and

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \ . \tag{3.9b}$$

Here $F(\omega)$ is the transform of $f(t)$ and ω is the frequency. The second relation is the inverse transformation. From eq. (3.9b) one sees that differentiation in time is equivalent to multiplication by $i\omega$ in the frequency domain, or

$$f(t) \leftrightarrow F(\omega)$$

and

$$\frac{\partial}{\partial t} \leftrightarrow (i\omega) \ .$$

One may transform eqs. (3.5, 6, 7 and 8) as follows:

$$(\nabla^2+k_L^2)\nabla\cdot\vec{u}_L = -\frac{1}{c_L^2}\nabla\cdot\vec{f}_L \tag{3.5c}$$

$$[\nabla x\nabla x - k_T^2](\nabla x\vec{u}_T) = \frac{1}{c_T^2}(\nabla x\vec{f}_T) \tag{3.6c}$$

$$[\nabla^2 + k_L^2]\vec{u}_L = -\frac{1}{c_L^2}\vec{f}_L \tag{3.7a}$$

$$[\nabla x\nabla x - k_T^2]\vec{u}_T = \frac{1}{c_T^2}\vec{f}_T\ , \tag{3.8a}$$

where $k_L = \omega/c_L$, $k_T = \omega/c_T$ are the longitudinal and transverse wave-numbers, respectively. The field quantities $\vec{u}$ and $\vec{f}$ are now understood to be functions of ω instead of time t, or

$$\vec{u} = \vec{u}(\vec{x},\ \omega)$$

and

$$\vec{f} = \vec{f}(\vec{x},\ \omega)\ .$$

Solutions to the Homogeneous Vector Equations

We write the homogeneous forms for the transformed field equations by setting the body force $\vec{f}$ to zero. In general, no body forces act in a simple thermo-mechanical field. Later, we introduce a field excitation in the liquid medium by considering $\vec{f}$ to be concentrated in a small region of space. The solid medium, however, does not have any direct excitation, so the field is described by homogeneous differential forms in space. We reduce eqs. (3.5c), (3.6c), (3.7a) and (3.8a) to the following:

$$(\nabla^2 + k_L^2)\nabla\cdot\vec{u}_L = 0 \tag{3.10}$$

$$(\nabla^2 + k_L^2)\vec{u}_L = 0 \tag{3.10a}$$

$$(\nabla x\nabla x - k_T^2)(\nabla x\vec{u}_T) = 0 \tag{3.11}$$

$$(\nabla x\nabla x - k_T^2)\vec{u}_T = 0 \tag{3.11a}$$

Solutions to eqs. (3.10a) and (3.11a) may be written as in the classical electromagnetic theory [32] as

$$\vec{u}_L = \nabla\phi_L \tag{3.12}$$

and

$$\vec{u}_T = \nabla \times \vec{A} \ , \tag{3.13}$$

where ϕ_L is a scalar potential and $\vec{A}$ is a vector potential. One usually imposes on the vector potential the following condition

$$\nabla \cdot \vec{A} = 0 \tag{3.14}$$

to eliminate the possibility of a component of $\vec{A}$ being the gradient of another scalar function. If one writes

$$\vec{A} = \vec{A}' + \nabla\Phi \quad ,$$

where $\nabla \cdot \vec{A}' = 0$, the divergence of $\vec{A}$ becomes

$$\nabla \cdot \vec{A} = \nabla^2\Phi \ .$$

The $\nabla^2\Phi$ term does not contribute to the solution, eq. (3.13), since curl grad $\Phi = 0$. The condition (3.14) eliminates this ambiguity.

Turning to the longitudinal field for the present, we rearrange eq. (3.10a)

$$\vec{u}_L = - \frac{\nabla(\nabla \cdot \vec{u})}{k_L^2} \quad . \tag{3.10b}$$

Substituting eq. (3.12) into the left-hand side of this gives

$$\nabla(\phi_L + \frac{1}{k_L^2} \nabla \cdot \vec{u}_L) = 0 \quad . \tag{3.15}$$

Integrating this result gives

$$\phi_L(\vec{x}, \omega) = - \frac{1}{k_L^2} (\nabla \cdot \vec{u}_L) + \phi_0(\omega) \quad , \tag{3.15a}$$

where $\phi_0(\omega)$ is uniform spatially, and may be set to zero with no loss in generality because we want solutions that vary spatially. Substituting this result into eq. (3.10) shows that ϕ_L satisfies the following scalar Helmholtz equation:

$$(\nabla^2 + k_L^2)\phi_L = 0 \quad . \tag{3.16}$$

One sees that the longitudinal field is the gradient of a scalar function which is obtained by solving a Helmholtz equation.

Looking at the transverse field, we see that the vector potential $\vec{A}$ is a solution of eq. (3.11a) provided $\nabla\cdot\vec{A} = 0$, or

$$(\nabla x \nabla x - k_T^2)\vec{A} = 0.$$

This shows that if a transverse vector satisfying eq. (3.11a) can be found, the curl of the vector is also a solution. This verifies that eq. (3.13) is a solution to eq. (3.11a).

We wish to find expressions for the vector potential $\vec{A}$. To proceed further, we must specify the coordinate system to be used. A cylindrical (r,z,θ) system is used, where z is the vertical axis. This is the most convenient since the boundaries lie in the $(r-\theta)$ plane [Figure (1)], and we expect to have symmetry in the polar coordinate θ due to the type of field excitation.

Following the discussion in Morse and Feshbach [32], we express the vector potential as

$$\vec{A}_{HS} = k_T \hat{e}_z \phi_{HS} \quad , \tag{3.17}$$

where ϕ_{HS} is a scalar function. The subscript HS denotes this solution as a "horizontal shear" (transverse) wave. We see from eq. (3.13) that the displacement of the HS polarization is in the $r-\theta$ (horizontal) plane. The condition (3.14) requires that

$$\nabla\cdot\vec{A}_{HS} = k_T \frac{\partial\phi_{HS}}{\partial z} = 0,$$

or

$$\phi_{HS} = \phi_{HS}(r, \theta, \omega). \tag{3.17a}$$

From the discussion above, we know that if the form (3.17) is a solution to the transverse field, the curl of it is also, or

$$\vec{A}_{VS} = \nabla x(\hat{e}_z\phi_{VS}), \tag{3.18}$$

where ϕ_{VS} is another scalar potential function. The VS designation implies that the solution is a "vertical shear" wave. This is evident due to the curl operation. The k_T factor has been introduced in eq. (3.17) to give ϕ_{HS} and ϕ_{VS} the same dimensions.

We need only verify that solution (3.17) satisfies the field equation. We substitute eq. (3.17) into eq. (3.11a), using condition (3.14) to give

$$(\nabla^2 + k_T^2)\phi_{HS} = 0, \tag{3.19}$$

where eq. (2.49) has been used. The scalar function for the HS polarization must satisfy the Holmholtz operator (3.19) if (3.17) is to be a solution to the transverse field. The same condition must apply to the VS polarization if it is to be a solution, or

$$(\nabla^2 + k_T^2)\phi_{VS} = 0 . \tag{3.19a}$$

To summarize, we have reduced the elastic field equation into transverse and longitudinal polarizations. Solutions for both have been obtained in terms of scalar functions satisfying Helmholtz equations. The transverse field has two polarizations denoted as HS and VS waves. The displacement field may be written as

$$\vec{u} = \vec{u}_L + \vec{u}_{HS} + \vec{u}_{VS}$$

where, from eqs. (3.12), (3.17) and (3.18) we have

$$\begin{aligned} \vec{u}_L &= \nabla\phi_L, \\ \vec{u}_{HS} &= k_T \nabla x \hat{e}_z \phi_{HS} , \\ \vec{u}_{VS} &= \nabla x (\nabla x \hat{e}_z \phi_{VS}) \end{aligned} \tag{3.20}$$

and the potentials ϕ_L, ϕ_{HS} and ϕ_{VS} satisfy

$$(\nabla^2 + k_L^2)\phi_L = 0$$

and $$(\nabla^2 + k_T^2)\begin{bmatrix}\phi_{HS} \\ \phi_{VS}\end{bmatrix} = 0 .$$

In addition, $\phi_{HS} = \phi_{HS}(r,\theta,\omega)$. The problem has been reduced to finding solutions of scalar partial differential equations.

B. The Viscoelastic Medium

We wish to analyze the field equation for the viscoelastic medium given by

$$\rho_0 \partial_t^2 \vec{u} - [(\lambda+2\mu) + (\lambda'+2\mu')\partial_t]\nabla(\nabla\cdot\vec{u}) + [\mu+\mu'\partial_t]\nabla x(\nabla x\vec{u}) = \rho_0 \vec{f} \tag{2.50}$$

One notes that this differs from the relation for the elastic solid only by the addition of the damping terms (first-order time derivatives). We may separate this field equation into longitudinal and transverse parts as was done for the elastic solid. First, we take the Fourier transform of eq. (2.50)

$$\rho_0 \omega^2 \vec{u} + [(\lambda+2\mu) + i\omega(\lambda'+2\mu')]\nabla(\nabla\cdot\vec{u}) + -(\mu+i\omega\mu')\nabla x(\nabla x\vec{u}) = -\rho_0 \vec{f} \tag{3.21}$$

We take the divergence and curl of (3.21) to obtain

$$\{\rho_0\omega^2 + [(\lambda+2\mu) + i\omega(\lambda'+2\mu')]\nabla^2\}(\nabla\cdot\vec{u}) = -\rho_0(\nabla\cdot\vec{f}) \tag{3.22}$$

and

$$[\rho_0\omega^2 - (\mu+i\omega\mu')\nabla x\nabla x](\nabla x\vec{u}) = -\rho_0(\nabla x\vec{f}) \quad . \tag{3.23}$$

Introducing c_L and c_T and damping coefficients b_L and b_T defined as

$$b_L = \frac{\lambda'+2\mu'}{\rho c_L^2} \quad , \quad b_T = \frac{\mu'}{\rho c_T^2} \quad .$$

Eqs. (3.22) and (3.23) may be written

$$\{(1 + i\omega b_L)\nabla^2 + \frac{\omega^2}{c_L^2}\}(\nabla\cdot\vec{u}) = -\frac{1}{c_L^2}(\nabla\cdot\vec{f}) \tag{3.22a}$$

and $$\{-(1 + i\omega b_T)\nabla x\nabla x + \frac{\omega^2}{c_L^2}\}(\nabla x\vec{u}) = -\frac{1}{c_T^2}(\nabla x\vec{f}) \quad . \tag{3.23a}$$

In the solid medium the body force $\vec{f}$ does not arise, so we write these equations in homogeneous form

$$(\nabla^2 + k_L^2)(\nabla\cdot\vec{u}) = 0 \tag{3.24}$$

$$(-\nabla x\nabla x + k_T^2)(\nabla x\vec{u}) = 0, \tag{3.25}$$

where $$k_L^2 = \frac{\omega^2}{c_L^2(1+i\omega b_L)}$$

and $$k_T^2 = \frac{\omega^2}{c_T^2(1+i\omega b_T)} \quad .$$

Equations (3.24) and (3.25) are Helmholtz operators. The field equations are of the same form as eqs. (3.10) and (3.11) for the elastic solid. We may write, analogous to eq. (3.10a) and (3.11a) the following for the viscoelastic medium:

$$(\nabla^2 + k_L^2)\vec{u}_L = 0$$

and $$(\nabla x \nabla x - k_T^2)\vec{u}_T = 0 \ .$$

The introduction of viscoelasticity does not change the form of the field equations in the frequency domain. The only effect is to change the wavenumbers to complex quantities. We see from the form of the wavenumbers given in eq. (3.25) that they are in the second and fourth quadrants of the complex plane. We may use the same solutions for the field obtained in the previous section, eqs. (3.12), (3.17) and (3.18), where the potential functions ϕ_L, ϕ_{HS} and ϕ_{VS} satisfy Helmholtz operators

$$(\nabla^2 + k_L^2)\phi_L = 0$$

$$(\nabla^2 + k_T^2)\begin{bmatrix}\phi_{HS} \\ \phi_{VS}\end{bmatrix} = 0,$$

where now the k_L and k_T are complex quantities instead of real numbers.

C. <u>Field Excitation in the Liquid</u>

The field equation for the inviscid liquid can be written from eqs. (2.46) and (2.34a) as

$$\rho_0 \partial_t^2 \vec{u} - \lambda \nabla (\nabla \cdot \vec{u}) - \rho_0 \vec{f}. \qquad (3.26)$$

This equation can be obtained as a special case of the elastodynamic equation (3.1) by setting the rigidity μ to zero. This approach was shown to be consistent with the conventional one used in fluid mechanics where the stress tensor is expressed in terms of the pressure [eq. (2.47)].

If one takes the divergence of eq. (3.26), one obtains

$$[\nabla^2 - \frac{1}{c_0^2}\partial_t^2] \quad (\nabla \cdot \vec{u}) = - \frac{\nabla \cdot \vec{f}}{c_0^2} \ , \qquad (3.26a)$$

where c_0 is the adiabatic sound velocity given as

$$c_0 = \left[\left(\frac{\partial p}{\partial \rho}\right)_s \right]^{\frac{1}{2}}$$

or

$$c_0 = \frac{\lambda}{\rho_0} = \frac{K}{\rho_0} .$$

Now we separate the vector fields $\vec{u}$ and $\vec{f}$ into longitudinal and transverse parts. The body force $\vec{f}$ must be longitudinal in the liquid, or

$$\vec{f} = \vec{f}_L$$

and

$$\vec{f}_T = 0. \tag{3.27}$$

Taking the curl of eq. (3.26) then requires

$$\nabla x \vec{u} = \nabla x \vec{u}_T = 0. \tag{3.28}$$

The dynamic field in the liquid is then purely longitudinal, or

$$\vec{u} = \vec{u}_L. \tag{3.29}$$

We may write eq. (3.26a) as

$$[\nabla^2 - \frac{1}{c_0^2} \partial_t^2] \ (\nabla \cdot \vec{u}_L) = - \frac{\nabla \cdot \vec{f}_L}{c_0^2} \tag{3.26a}$$

or, in the frequency domain

$$(\nabla^2 + k_0^2)(\nabla \cdot \vec{u}_L) = - \frac{\nabla \cdot \vec{f}_L}{c_0^2} , \tag{3.26b}$$

where $k_0 = \omega / c_0$. Now, since the divergence of a vector is a scalar, we may write, in the frequency domain,

$$\nabla \cdot \vec{u}_L = -k_0^2 \ \phi_0$$

and

$$\nabla \cdot \vec{f}_L = -k_0^2 \ \psi_0 \ , \tag{3.30}$$

where ϕ_0 and ψ_0 are scalar functions for the fluid's displacement and body force. The constant factors in eq. (3.30) were introduced to make the ϕ_0 potential dimensionally consistent with the longitudinal potential function ϕ_L for the elastic solid given in eq. (3.15a). Substitution of eq. (3.30) into the field equation (3.26b) gives

$$(\nabla^2 + k_0^2)\phi_0 = -\frac{\psi_0}{c_0^2}\ . \tag{3.31}$$

This result is seen to be an inhomogeneous scalar Helmholtz equation. The excitation to the field (liquid overlying a layered solid) is taken in the liquid. The excitation must represent the effect of an acoustic transducer as a sound source. We wish to represent this sound source in terms of the body force $\vec{f}_L$ or its divergence. Now, the transducer is small relative to the other dimensions of the acoustic field. We may consider the sound source and the body force field to be located in a small spherical region of radius a. If one integrates the divergence of $\vec{f}$ over the volume V of the sphere, one obtains using Gauss' theorem:

$$\int_V (\nabla\cdot\vec{f})dV = \oiint_S \vec{f}\cdot\hat{n}dS. \tag{3.32}$$

If one assumes that the force field $\vec{f}$ acts in a direction normal to the surface of the spherical volume, or radially outward, we may write the radial component of $\vec{f}$ as f_r . If, furthermore, we assume f_r is constant, one has

$$\int_V (\nabla\cdot\vec{f})dV = \oiint_S f_r dS = 4\pi a^2 (f_r)_{r=a}\quad . \tag{3.32a}$$

Equation (3.32a) implies that the divergence of $\vec{f}$ may be represented by a Dirac delta function $\delta(\vec{r}-\vec{r}\,')$, where $\vec{r}$ is the field point and $\vec{r}\,'$ is the source point. We recall the characteristics of the delta function [54]

$$\left.\begin{aligned} &\int_V \delta(\vec{r}-\vec{r}')dV = 1 \\ &\text{if } \vec{r}' \text{ is in V and} \\ &\delta(\vec{r}-\vec{r}') = 0 \quad \text{for } \vec{r} \neq \vec{r}'. \end{aligned}\right\} \tag{3.33}$$

We note that the right-hand side of eq. (3.32a) is a constant, so from the integral representation of the delta function in (3.33) we see that $\nabla\cdot\vec{f}$ is proportional to the delta function. We also see from eq. (3.30) that the force potential ψ_0 is proportional to the delta function. We may write for the right-hand side of eq. (3.31) the following:

$$\frac{\psi_0(\vec{r},\omega)}{c_0^2} = H(\omega)\delta(\vec{r}-\vec{r}'), \tag{3.34}$$

where $H(\omega)$ is the transformed from of the time dependence of the field excitation. That is, eq. (3.34) separates the spatial dependence of ψ_0 from the time or frequency dependence. The inverse transform of $H(\omega)$ will be taken as $h(t)$. Now, one may denote the response of the field ϕ_0 due to the excitation as a Green's function [34], [50] denoted in the frequency domain as

$$G(\vec{r}, \vec{r}', \omega). \tag{3.35}$$

Applying eqs. (3.34) and (3.35) to (3.31) gives for the liquid field

$$(\nabla^2 + k_0^2)G(\vec{r},\vec{r}',\omega) = -H(\omega)\delta(\vec{r}-\vec{r}'). \tag{3.36}$$

Denoting the Green's function in the time domain as

$$g(\vec{r},\vec{r}',t)$$

enables us to write eq. (3.32) in the time domain as follows:

$$(\nabla^2 - \frac{1}{c_0^2}\partial_t^2)g(\vec{r},\vec{r}',t) = -h(t)\delta(\vec{r}-\vec{r}') \tag{3.37}$$

Equation (3.36) is much easier to solve for the present problem, so this form will be used in the following chapters. After solving for the Green's function in the frequency domain, we obtain the time domain representation by taking the inverse Fourier transform (3.9b)

$$g(\vec{r},\vec{r}',t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\vec{r},\vec{r}',\omega)e^{i\omega t}\,d\omega \quad . \tag{3.38}$$

The field excitation as given by eq. (3.36) occurs only at a point $\vec{r}'$. From the definition of the delta function (3.33), one sees that eq. (3.36) reduces to

$$(\nabla^2 + k_0^2)G(\vec{r},\vec{r}',\omega) = 0 \quad \text{for } \vec{r} \neq \vec{r}'. \tag{3.39}$$

That is, the governing equation is a homogeneous form except at $\vec{r} = \vec{r}'$. The displacement field in the liquid may be expressed in the same form as for the solid, or

$$\vec{u}_0 = \nabla G \quad \text{for } \vec{r} = \vec{r}', \tag{3.40}$$

from eq. (3.12) for the elastic solid. This is a solution to the field equation (3.39) due to the form taken for $\nabla\cdot\vec{u}$ in eq. (3.30).

The location of the source point $\vec{r}'$ will be taken on the z-axis in a cylindrical coordinate system (r,z,θ). The type of excitation is obviously symmetric with respect to the angular coordinate θ, so the field will be a function of only z and r. Furthermore, one would not expect this source representation to excite the HS polarization [eq. (3.17)]. The HS polarization consists of motion in the r-θ plane only, and does not vary with z. Therefore, the dilatational nature of the acoustic or liquid field excitation cannot impart this type of motion. The motion in the solid field will then be restricted to longitudinal and VS polarizations. In addition, the θ-symmetry in the liquid will apply as well to the solid.

D. The Stress and Displacement Fields

We wish to express the stress and displacement fields in terms of the scalar potential functions ϕ_L, ϕ_{HS} and ϕ_{VS}. Due to the symmetry in the field, we use cylindrical polar coordinates (r,z,θ) where z is the vertical coordinate. The displacement has been expressed in terms of the potential functions in eqs. (3.20) as

$$\vec{u}_L = \nabla\phi_L,$$

$$\vec{u}_{HS} = k_T \nabla x(\hat{e}_z \phi_{HS}) \tag{3.20}$$

and $$\vec{u}_{VS} = \nabla x(\nabla x \hat{e}_z \phi_{VS}) \ ,$$

where $\vec{u} = \vec{u}_L + \vec{u}_{HS} + \vec{u}_{VS}$. Using (2.49), the expression for $\vec{u}_{VS}$ may be written

$$\vec{u}_{VS} = \nabla(\nabla \cdot \hat{e}_z \phi_{VS}) - \nabla^2(\hat{e}_z \phi_{VS})$$

or $$\vec{u}_{VS} = \nabla(\frac{\partial\phi_{VS}}{\partial z}) - \hat{e}_z \nabla^2 \phi_{VS} \ . \tag{3.41}$$

To compute the expressions for the components of the displacement vector, we recall some vector identities for cylindrical coordinates. Taking f as a scalar and $\vec{B}$ as a vector field, we have:

$$\nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z$$

$$\nabla \cdot \vec{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\theta}{\partial\theta} + \frac{\partial B_z}{\partial z}$$

$$\nabla x \vec{B} = (\frac{1}{r}\frac{\partial B_z}{\partial\theta} - \frac{\partial B_\theta}{\partial z})\hat{e}_r + (\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r})\hat{e}_\theta + \frac{1}{r}[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial B_r}{\partial\theta}]\hat{e}_z$$

and

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial r}{\partial f}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2} \ ,$$

where B_r, B_θ and B_z are the r, θ and z components of $\vec{B}$, and $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_z$ are the corresponding unit vectors.

We may write the displacement vector as:

$$\vec{u} = u_r\hat{e}_r + u_\theta\hat{e}_\partial + u_z\hat{e}_z \quad .$$

Applying eqs. (3.42) to (3.20) and (3.41) gives, for the various polarizations of the displacement vector:

$$\vec{u}_L = \frac{\partial\phi_L}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi_L}{\partial\theta}\hat{e}_\theta + \frac{\partial\phi_L}{\partial z}\hat{e}_z,$$

$$\vec{u}_{HS} = k_T\frac{1}{r}\frac{\partial\phi_{HS}}{\partial\theta}\hat{e}_r - \frac{\partial\phi_{HS}}{\partial r}\hat{e}_\theta \tag{3.43}$$

and

$$\vec{u}_{VS} = \frac{\partial^2\phi_{VS}}{\partial r\partial z}\hat{e}_r + \frac{\partial^2\phi_{VS}}{\partial\theta\partial z}\hat{e}_\theta + \left(\frac{\partial^2}{\partial z^2} + k_T^2\right)\phi_{VS}\hat{e}_z.$$

The components of the displacement may be written

$$u_r = \frac{\partial\phi_L}{\partial r} + \frac{k_T}{r}\frac{\partial\phi_{HS}}{\partial\theta} + \frac{\partial^2\phi_{VS}}{\partial r\partial z},$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi_L}{\partial\theta} - k_T\frac{\partial\phi_{HS}}{\partial r} + \frac{1}{r}\frac{\partial^2\phi_{VS}}{\partial\theta\partial z} \tag{3.44}$$

and

$$u_z = \frac{\partial\phi_L}{\partial z} + \qquad + \left(\frac{\partial^2}{\partial z^2} + k_T^2\right)\phi_{VS}.$$

The displacement field simplifies considerably when θ-symmetry is present. It simplifies even more when the HS polarization does not occur. The field excitation discussed in the preceding section includes both of these. We write for the displacement field of interest

$$u_r = \frac{\partial\phi_L}{\partial r} + \frac{\partial^2\phi_{VS}}{\partial r\partial z},$$

$$u_\theta = 0 \tag{3.45}$$

and

$$u_z = \frac{\partial\phi_L}{\partial z} + \left(\frac{\partial^2}{\partial z^2} + k_T^2\right)\phi_{VS} \quad .$$

Equations (3.45) will be used to solve the boundary-value problem to be developed in the following chapter.

The simplified stress field is now developed. We recall the expression for the stress developed earlier

$$\sigma_{ij} = \lambda\varepsilon_{\ell\ell}\delta_{ij} + 2\mu\varepsilon_{ij}. \tag{2.42}$$

Viscoelasticity can be introduced easily by taking the Lamé parameters as complex constants in the frequency domain. Now, in eq. (2.42), $\varepsilon_{\ell\ell}$ is, in invariant notation, $\nabla\cdot\vec{u}$. From eq. (3.15a)

$$\varepsilon_{\ell\ell} = \nabla\cdot\vec{u} = -k_L^2\phi_L.$$

We write from eq. (2.42) the stress tensor in cylindrical coordinates

$$\begin{aligned}
\sigma_{rr} &= \lambda\nabla\cdot\vec{u} + 2\mu\varepsilon_{rr} \\
\sigma_{zz} &= \lambda\nabla\cdot\vec{u} + 2\mu\varepsilon_{zz} \\
\sigma_{\partial\partial} &= \lambda\nabla\cdot\vec{u} + 2\mu\varepsilon_{\partial\partial} \\
\sigma_{rz} &= 2\mu\varepsilon_{rz} \\
\sigma_{z\theta} &= 2\mu\varepsilon_{z\theta} \\
\sigma_{r\theta} &= 2\mu\varepsilon_{r\theta}.
\end{aligned} \tag{3.46}$$

The strain tensor in cylindrical coordinates is expressed in terms of the displacement as [26]:

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\varepsilon_{\partial\partial} &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} \\
2\varepsilon_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\
2\varepsilon_{z\theta} &= \frac{1}{r}\frac{\partial u_z}{\partial\theta} + \frac{\partial u_\theta}{\partial z} \\
2\varepsilon_{r\theta} &= \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} .
\end{aligned} \tag{3.47}$$

Introducing the field symmetries reduces eq. (3.47) to

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\varepsilon_{\theta\theta} &= \frac{u_r}{r} \\
2\varepsilon_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\
2\varepsilon_{z\theta} &= 0 \\
2\varepsilon_{r\theta} &= 0 \quad .
\end{aligned} \tag{3.47a}$$

We apply eqs. (3.45) to (3.47a) to give for the symmetric strain field:

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial^2}{\partial r^2} (\phi_L + \frac{\partial}{\partial z} \phi_{VS}) \\
\varepsilon_{zz} &= \frac{\partial^2 \phi_L}{\partial z^2} + \frac{\partial}{\partial z} (\frac{\partial^2}{\partial z^2} + k_T^2)\phi_{VS} \\
\varepsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial}{\partial r}(\phi_L + \frac{\partial}{\partial z} \phi_{VS}) \\
2\varepsilon_{rz} &= 2 \frac{\partial^2}{\partial r \partial z} \phi_L + \frac{\partial}{\partial r} (2 \frac{\partial^2}{\partial z^2} + k_T^2)\phi_{VS} \\
2\varepsilon_{z\theta} &= 0 \\
2\varepsilon_{r\theta} &= 0 \quad .
\end{aligned} \tag{3.47b}$$

We apply these results to the stress field (3.46), expressing $\nabla \cdot \vec{u}$ in terms of the longitudinal potential to give

$$\begin{aligned}
\sigma_{rr} &= -\lambda k_L^2 \phi_L + 2\mu \frac{\partial^2}{\partial r^2} (\phi_L + \frac{\partial}{\partial z} \phi_{VS}) \\
\sigma_{zz} &= -\lambda k_L^2 \phi_L + 2\mu \frac{\partial}{\partial z} [\frac{\partial \phi_L}{\partial z^2} + k_T^2)\phi_{VS}] \\
\sigma_{\theta\theta} &= -\lambda k_L^2 \phi_L + 2\mu \frac{1}{r}\frac{\partial}{\partial r}(\phi_L + \frac{\partial}{\partial z} \phi_{VS})
\end{aligned} \tag{3.48}$$

$$\sigma_{rz} = \mu \frac{\partial}{\partial r}[2 \frac{\partial \phi_L}{\partial z} + (2 \frac{\partial^2}{\partial z^2} + k_T^2)\phi_{VS}]$$

$$\sigma_{z\theta} = 0 \tag{3.48}$$

$$\sigma_{r\theta} = 0.$$

Equations (3.45) and (3.48) are the required expressions for the displacement and stress fields. These expressions will be used to evaluate boundary conditions at interfaces between different media.

IV. THE SOLUTION OF THE BOUNDARY-VALUE PROBLEM

A. Preliminary

We wish to derive an expression for the response of the system shown schematically in Figure 1. The system is a field consisting of a liquid layer overlying a layered solid halfspace. We place an acoustic source in the liquid and use the linearized viscoelastic model for the solid. The precise geometry and coordinate system used for the problem is shown in Figure 2. The source is located on the z-axis of a cylindrical coordinate system. The symmetry of the source eliminates the polar (θ) dependence and HS polarizations in the solid are ruled out.

The problem is posed as follows. We wish to solve the governing equation for the acoustic field

$$(\nabla^2 + k_0^2)G(\vec{r},\vec{r}\,',\omega) = -H(\omega)\delta(\vec{r}-\vec{r}\,') \tag{3.36}$$

subject to appropriate boundary conditions at the surface of the liquid layer and at the liquid-solid interface. Now, the acoustic field is coupled mechanically to the viscoelastic field, so we must solve two Helmholtz equations for each solid layer

$$(\nabla^2 + k_{Lj}^2)\phi_{Lj} = 0 \tag{3.16}$$

$$(\nabla^2 + k_{Tj}^2)\phi_{VSj} = 0, \quad j = 1, 2, \ldots n, \tag{3.19a}$$

where the subscript j refers to the particular layer. The problem will be solved by introducing a Fourier-Bessel transform [55] to the governing equations (3.36), (3.16) and (3.19a). Then boundary conditions will be applied at the interfaces between each layer and at the surface of the liquid, in accordance with the results of Chapter II. We begin by

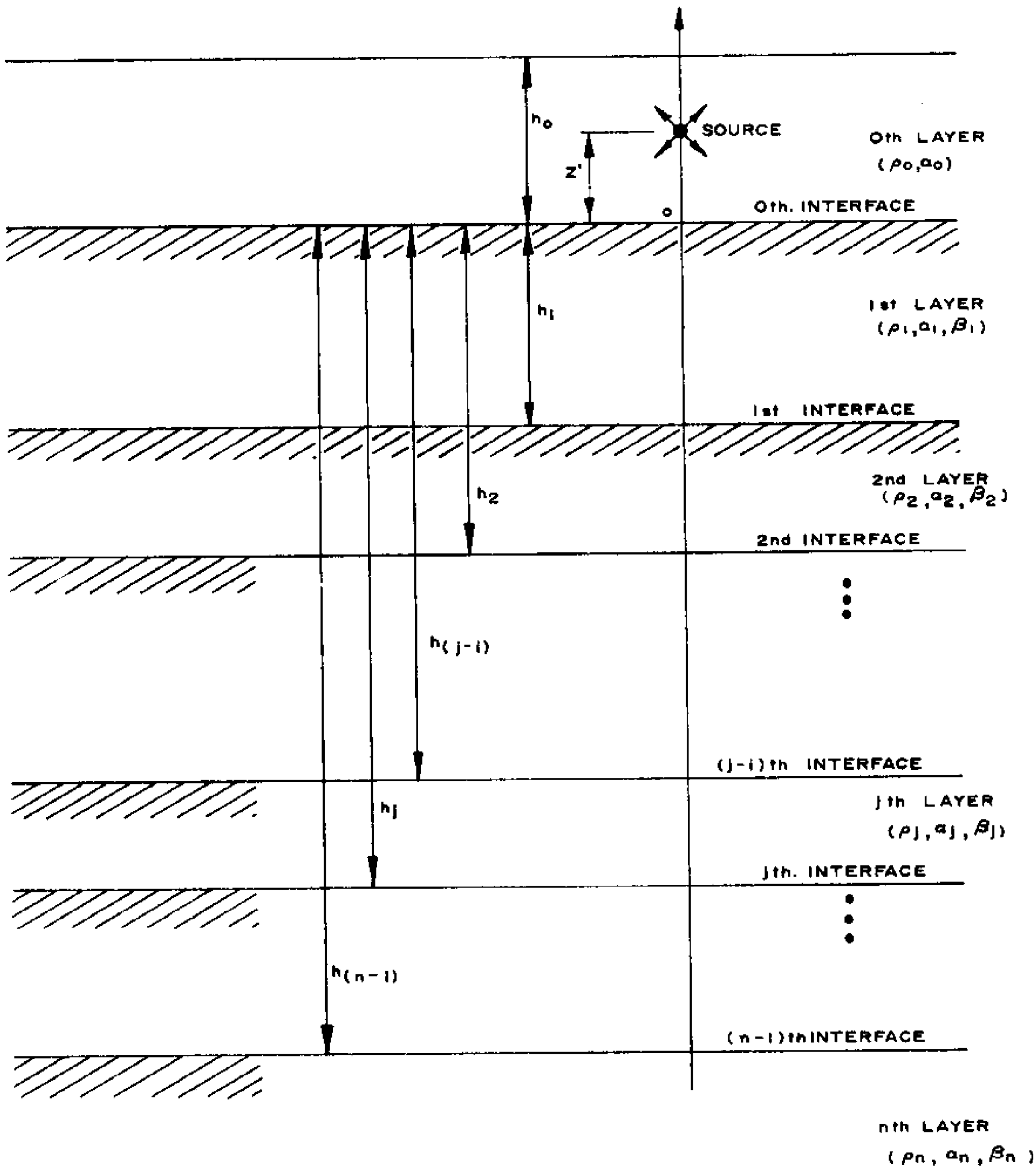


FIGURE 2

Dimensions and Coordinate System Used for Multi-Layer Problem solving eq. (3.36). First we express the Dirac delta function as [10]

$$\delta(\vec{r}-\vec{r}\,') = \frac{\delta(r)\delta(z-z')}{2\pi r} . \tag{4.1}$$

This form implies that the source is at $r = 0$, $z = z'$, as shown in Figure 2. The Green's function may be expressed in the form

$$G(\vec{r},\vec{r}\,',\omega) = G(r,z,z',\omega). \tag{4.2}$$

The ∇^2 operator in eq. (3.36) reduces from (3.41) to the form

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{\partial^2}{\partial z^2} .$$

Applying this result and eqs. (4.1) and (4.2) to eq. (3.36) gives

$$[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{\partial^2}{\partial z^2} + k_0^2]G(r,z,z',\omega) = -\frac{\delta(r)\delta(z-z')}{2\pi r} H(\omega). \tag{4.3}$$

We apply the Fourier-Bessel transform to eq. (4.3) to eliminate the variable r as a differential form. The Fourier-Bessel transform pair is written as [51]

$$\left.\begin{aligned} \underline{g}(\zeta) &= \int_0^\infty g(r)J_0(\zeta r)r\,dr \\ g(r) &= \int_0^\infty \underline{g}(\zeta)J_0(\zeta r)\zeta\,d\zeta, \end{aligned}\right\} \tag{4.4}$$

where the bar under the variable denotes the transformed quantity. The zeroth order Bessel function has been used in eq. (4.4) due to the θ-symmetry. Following a related calculation for a ring source excitation from Stakgold [51], we transform eq. (4.3) using eq. (4.4) to obtain

$$\int_0^\infty \frac{1}{r}\frac{\partial}{\partial r}[r\frac{\partial}{\partial r} G(r,z,z',\omega)]r\,dr + (\frac{d^2}{dz^2} + k_0^2)\underline{G}(\zeta,z,z',\omega) =$$

$$= \frac{-\delta(z-z')H(\omega)}{2\pi}\int_0^\infty \delta(r)J_0(\zeta r)dr, \tag{4.5}$$

where $\underline{G}$ is the transformed form of the Green's function.

Due to the sifting property of the delta function, the right-hand side of eq. (4.5) reduces to

$$\frac{-\delta(z-z')}{2\pi}\ , \text{ since } J_0(0) = 1.$$

We apply partial integration twice to the first term on the left-hand side of eq. (4.5), giving as a result

$$\left[\frac{d^2}{dz^2} + (k_0^2 - \zeta^2)\right]\underline{G}(\zeta, z, z', \omega) = -\frac{\delta(z-z')}{2\pi} H(\omega) \tag{4.6}$$

Equation (4.6) is an ordinary differential equation in z. The r-dependence in eq. (4.3) has been reduced to an algebraic form in the transform domain.

We solve eq. (4.6) using a Formal Green's function procedure given in Hildebrand [14]. We write two independent solutions to the homogeneous form of eq. (4.6) for the Green's function above and below the source:

$$\left[\frac{d^2}{dz^2} + (k_0^2 - \zeta^2)\right]\underline{G}_{>}(\zeta, z, z') = 0 \quad \text{for } z' < z < h_0$$

and

$$\left[\frac{d^2}{dz^2} + (k_0^2 - \zeta^2)\right]\underline{G}_{<}(\zeta, z, z') = 0 \quad \text{for } 0 < z < z', \tag{4.7}$$

where $\underline{G}_{>}$ denotes the Green's function above the source and $\underline{G}_{<}$ below the source. Solutions to eq. (4.7) are written

$$\underline{G}_{>} = Pe^{-a_0 z} + Qe^{a_0 z} \quad \text{for } z' < z < h_0$$

and

$$\underline{G}_{<} = Re^{-a_0 z} + Se^{a_0 z} \quad \text{for } 0 < z < z', \tag{4.8}$$

where $a_0 = (\zeta^2 - k_0^2)^{1/2}$, and the positive square root is taken. The quantities P, Q, R and S are functions of ζ and the physical parameters. These must be evaluated using the boundary conditions of the problem.

In addition, two requirements are imposed by the formal Green's function procedure. These are the continuity of $\underline{G}$ at the source ($z=z'$) and the jump in the first derivative of $\underline{G}$ at the source. The first condition requires

$$\underline{G}_{>}(\zeta,z',z') = \underline{G}_{<}(\zeta,z',z'). \tag{4.9a}$$

The second condition arises from integrating eq. (4.6) across the source in the z-direction:

$$\int_{z'-\varepsilon}^{z'+\varepsilon} \frac{d}{dz}\left(\frac{d\underline{G}}{dz}\right) dz = -\frac{H(\omega)}{2\pi},$$

where ε is taken as small, and the sifting property of $\delta(z-z')$ has been used. This result may be written

$$\left[\frac{d}{dz}\underline{G}_{>}(\zeta,z,z') - \frac{d}{dz}\underline{G}_{<}(\zeta,z,z')\right]_{z=z'} = -\frac{H(\omega)}{2\pi}. \tag{4.9b}$$

We may immediately impose one simple boundary condition. The pressure must vanish at the surface of the liquid. From eq. (3.48), we see that the stress tensor for the liquid above the source is

$$\begin{aligned} \sigma_{rr} &= \sigma_{\theta\theta} = \sigma_{zz} = -\lambda k_0^2\,\underline{G}_{>} \\ \sigma_{r\theta} &= \sigma_{rz} = \sigma_{\theta z} = 0 \quad , \end{aligned} \tag{4.10}$$

and from (2.47) we see the stress is related to the pressure by

$$\sigma_{ij} = -p'\delta_{ij} .$$

This implies that the Green's function vanishes at the liquid surface, or

$$\underline{G}_{>}(\zeta,z,z') = 0 \quad \text{at } z = h_0 . \tag{4.11}$$

Now we apply the three conditions (4.9a), (4.9b) and (4.11) to the solution form (4.8) to eliminate three of the unknown functions of ζ, leaving

$$
\begin{aligned}
G_{>} &= -2Q(\zeta)e^{a_0 h_0} \sinh[a_0(h_0-z)] \\
G_{<} &+ -2\{Q(\zeta)e^{a_0 h_0} \sinh a_0(h_0-z) + \frac{H(\omega)}{4\pi a_0} \sinh[a_0(z'-z)]\}
\end{aligned}
\qquad (4.12)
$$

The function $Q = Q(\zeta,z,z')$ will be evaluated by applying boundary conditions at the bottom of the liquid layer.

B. Green's Function for the Unbounded Liquid

We develop here a special case of the problem where the fluid is of infinite extent. The depth h_0 is infinite, and no solid bottom is present. Solutions to eq. (4.7) must be written in the form

$$
\left.
\begin{aligned}
G_{>} &= Pe^{-a_0 z} \quad \text{for } z > z' \\
&\text{and} \\
G_{<} &= Se^{a_0 z} \quad \text{for } z < z'.
\end{aligned}
\right\}
\qquad (4.13)
$$

The result is a special case of eq. (4.8). The terms with the Q and R factors were dropped so that the solutions vanish for $|z|$ large. This is equivalent to a radiation condition. We apply conditions (4.9a) and (4.9b) to the solutions giving

$$
\begin{aligned}
Pe^{-a_0 z'} &= Se^{a_0 z'} \\
&\text{and} \\
Pe^{-a_0 z'} + Se^{a_0 z'} &= \frac{H(\omega)}{2\pi a_0} .
\end{aligned}
\qquad (4.14)
$$

The solutions become

$$
\left.
\begin{aligned}
G_{>} &= \frac{H(\omega)e^{-a_0(z-z')}}{4\pi a_0} \quad \text{for } z > z' \\
&\text{and} \\
G_{<} &= \frac{H(\omega)e^{-a_0(z'-z)}}{4\pi a_0} \quad \text{for } z < z'.
\end{aligned}
\right\}
\qquad (4.15)
$$

We may combine the two expressions (4.15) into one

$$\underline{G}(\zeta,z,z',\omega) = \frac{H(\omega)}{4\pi}\,\frac{e^{-a_0(z_>-z_<)}}{a_0}\,, \tag{4.16}$$

where
$$z_> = \mathrm{Max}(z,z'),$$
and
$$z_< = \mathrm{Min}(z,z').$$

This notation is used to combine the two expressions $\underline{G}_>$ and $\underline{G}_<$ into one. It implies reciprocity. That is, the Green's function's form remains the same if the source point and the field point are interchanged. We write eq. (4.16) in the r-domain by taking the inverse transform using eq. (4.4)

$$G(r,z,z',\omega) = \frac{H(\omega)}{4\pi}\int_0^\infty \frac{e^{-a_0(z_>-z_<)}}{a_0}\,J_0(\zeta r)\zeta\,d\zeta\,, \tag{4.17}$$

where we recall that $a_0 = (\zeta^2-k_0^2)^{1/2}$. This result is the same as Sommerfeld's [51], except that we have a more general time dependence due to the $H(\omega)$ factor. Sommerfeld took a harmonic time dependence. The 4π factor occurs due to the form of eq. (3.36), and does not occur in Sommerfeld's result because of his different approach to the problem.

The Green's function for an unbounded medium is well known [46] [35]. We write

$$G(r,z,z',\omega) = \frac{H(\omega)}{4\pi}\,\frac{e^{-ik_0R}}{R}\,, \tag{4.18}$$

where $R = [(z-z')^2 + r^2]^{1/2}$ is the distance from source to receiver. One may write eq. (4.18) in the time domain using the inverse Fourier transform in time [eq. (3.38)] to obtain

$$g(r,z,z',t) = \frac{1}{8\pi^2 R}\int_{-\infty}^{\infty} H(\omega)e^{-ik_0R}\,e^{i\omega t}\,d\omega.$$

Recalling that $k_0 = \omega/c_0$, one has

$$g(r,z,z',t) = \frac{1}{8\pi^2 R}\int_{-\infty}^{\infty} H(\omega)e^{+i\omega(t-R/c_0)}\,d\omega\,. \tag{4.19}$$

Eq. (4.19) is a superposition of harmonic waves due to the kernel

$e^{i\omega(t-R/c_0)}$, where the spectrum $H(\omega)$ is a weighting function in the frequency domain. For a harmonic time dependence, the weighting function $H(\omega)$ becomes a Dirac delta function, or

$$H(\omega) = 2\pi\delta(\omega-\omega_0),$$

where ω_0 is the frequency of the signal. Eq. (4.19) reduces to the following for the harmonic time dependence

$$g(r,z,z',t) = \frac{1}{4\pi R}\, e^{i\omega_0(t-R/c_0)} \quad . \tag{4.20}$$

This result is a harmonic, steady-state spherical wave train propagating radially outward.

$$\underline{\phi}_{\alpha j} = \frac{1}{4\pi a_{\alpha j}} \left[A_j e^{-a_{\alpha j} z} + B_j e^{a_{\alpha j} z}\right]$$

and

$$\underline{\phi}_{\beta j} = \frac{1}{4\pi a_{\beta j}} \left[C_j e^{-a_{\beta j} z} + D_j e^{a_{\beta j} z}\right], \tag{4.22}$$

where

$$a_{\alpha j} = (\zeta^2 - k_{\alpha j}^2)^{1/2} ,$$

$$a_{\beta j} = (\zeta^2 - k_{\beta j}^2)^{1/2}$$

and $h_j < z < h_{(j-1)}$. The $a_{\alpha j}$ and $a_{\beta j}$ expressions represent a positive square root. We change notation in eq. (4.22), denoting longitudinal field quantities with an α and the VS field with a β subscript. The potentials $\underline{\phi}_{\alpha n}$ and $\underline{\phi}_{\beta n}$ for the n^{th} or last layer (a halfspace) must vanish as $-z \rightarrow \infty$. This requirement is met by setting the functions A_n and C_n in eq. (4.22) to zero. This is, in effect, a radiation condition. We may consider the B_j and D_j terms as representing outgoing waves propagating downward and the A_j and C_j terms as upward-traveling waves.

The solutions (4.22) may be applied to the expressions for the displacement field and the stress field eqs.[(3.45) and (3.48)]. Boundary conditions are then applied at the j^{th} interface separating the j^{th} and $(j+1)^{th}$ layer. The boundary conditions applicable to the solid-solid interface are:

1. Continuity of stress
2. Continuity of normal displacement
3. Continuity of tangential displacement

Expressing these three in the cylindrical coordinate system gives:

$$
\begin{aligned}
&\text{i)} \quad u_{zj} = u_{z(j+1)} \qquad \text{at } z = -h_j, \\
&\text{ii)} \quad u_{rj} = u_{r(j+1)} \qquad \text{"} \quad , \\
&\text{iii)} \quad \sigma_{rzj} = \sigma_{rz(j+1)} \qquad \text{"} \\
\text{and} \quad &\text{iv)} \quad \sigma_{zzj} = \sigma_{zz(j+1)} \qquad \text{"} \quad .
\end{aligned} \tag{4.23}
$$

Equations (4.23) may be expressed in terms of the unknown functions A_j, etc. in eq. (4.22) using eqs. (3.45) and (3.48). We write for convenience from eqs. (3.45) and (3.48):

$$
\begin{aligned}
u_r &= \frac{\partial}{\partial r} \left(\phi_\alpha + \frac{\partial \phi_\beta}{\partial z}\right) \quad , \\
u_z &= \frac{\partial \phi_\alpha}{\partial z} + \left(k_\beta^2 + \frac{\partial^2}{\partial z^2}\right)\phi_\beta, \\
\sigma_{zz} &= -\lambda k_\alpha^2 \phi_\alpha + 2\mu \frac{\partial}{\partial z} \left[\frac{\partial \phi_\alpha}{\partial z} + \left(\frac{\partial^2}{\partial z^2} + k_\beta^2\right)\phi_\beta\right] \\
\text{and} \quad \sigma_{rz} &= \mu \frac{\partial}{\partial r} \left[2 \frac{\partial \phi_\alpha}{\partial z} + \left(2 \frac{\partial^2}{\partial z^2} + k_\beta^2\right)\phi_\beta\right].
\end{aligned} \tag{4.24}
$$

We apply eqs. (4.24) to (4.23) using eqs. (4.22) to obtain a relation that may be written in matrix form as

$$
[a_j]\, \vec{A}_{j,j} = [a_{(j+1)}]\, \vec{A}_{(j+1),j} \quad , \tag{4.25}
$$

where the $[a_j]$ and $[a_{(j+1)}]$ factors are (4×4) matrices, each row corresponding to one of the boundary conditions (4.23). The vector quantities $\vec{A}_{j,j}$ and $\vec{A}_{(j+1),j}$ are (4×1) column matrices related to the unknown functions A_j, B_j, C_j and D_j in eq. (4.22). We denote the subscript (j+1) as j' for shorthand in the following. The $[a_j]$ matrix is given as

$$[a_j] = \begin{bmatrix} \rho_j\beta_j^2(2\zeta^2-k_{\beta j}^2) & \rho_j\beta_j^2(2\zeta^2-k_{\beta j}^2) & -2\rho_j\beta_j^2 a_{\beta j}\zeta^2 & 2\rho_j\beta_j^2 a_{\beta j}\zeta^2 \\ -2\rho_j\beta_j^2 a_{\alpha j} & 2\rho_j\beta_j^2 a_{\alpha j} & \rho_j\beta_j^2(2\zeta^2-k_{\beta j}^2) & \rho_j\beta_j^2(2\zeta^2-k_{\beta j}^2) \\ 1 & 1 & -a_{\beta j} & a_{\beta j} \\ -a_{\alpha j} & a_{\alpha j} & \zeta^2 & \zeta^2 \end{bmatrix}, \tag{4.26}$$

where $\beta_j^2 = \omega^2 k_{\beta j}^2$. The $[a_{(j+1)}] = [a_{j'}]$ matrix is obtained from (4.26) by increasing the index by one. We write the column matrix as follows:

$$\vec{A}_{j',j} = \begin{bmatrix} \dfrac{A_{j'} e^{a_{\alpha j'} h_j}}{a_{\alpha j'}} \\ \dfrac{B_{j'} e^{-a_{\alpha j'} h_j}}{a_{\alpha j'}} \\ \dfrac{C_{j'} e^{a_{\beta j'} h_j}}{a_{\beta j'}} \\ \dfrac{D_{j'} e^{-a_{\beta j'} h_j}}{a_{\beta j'}} \end{bmatrix}. \tag{4.27}$$

This relation may be written in a simpler form as the product of a column matrix and a diagonal matrix. We write

$$\vec{A}_{j',j} = [A_{j',j}]\vec{A}_{j'} , \tag{4.28}$$

where the column matrix $\vec{A}_{j'}$ is

$$\vec{A}_{j'} = \begin{bmatrix} \frac{A_{j'}}{a_{\alpha j'}} \\ \frac{B_{j'}}{a_{\alpha j'}} \\ \frac{C_{j'}}{a_{\beta j'}} \\ \frac{D_{j'}}{a_{\beta j'}} \end{bmatrix} , \tag{4.28a}$$

and the diagonal matrix is written as

$$[A_{j',j}] = \begin{bmatrix} e^{a_{\alpha j'} h_j} & 0 & 0 & 0 \\ 0 & e^{-a_{\alpha j'} h_j} & 0 & 0 \\ 0 & 0 & e^{a_{\beta j'} h_j} & 0 \\ 0 & 0 & 0 & e^{-a_{\beta j'} h_j} \end{bmatrix} . \tag{4.28b}$$

Substituting eq. (4.28) into (4.25) gives

$$[a_j][A_{j,j}]\vec{A}_j = [a_{j'}][A_{j',j}]\vec{A}_{j'} , \tag{4.29}$$

where we recall that $j' = (j+1)$. We solve the above for $\vec{A}_j$ by premultiplying both sides of eq. (4.29) by the inverse matrices $[a_j]^{-1}$ and $[A_{j,j}]^{-1}$ to obtain

$$\vec{A}_j = [A_{j,j}]^{-1}[a_j]^{-1}[a_{j'}][A_{j',j}]\vec{A}_{j'} . \tag{4.30}$$

To simplify the notation, we may write the product of the four square matrices in eq. (4.30) as one matrix relating $\vec{A}_j$ to $\vec{A}_{j'}$:

$$\vec{A}_j = [b_{(j+1),j}]\vec{A}_{(j+1)} , \tag{4.30a}$$

where
$$[b_{j',j}] = [A_{j,j}]^{-1}[a_j]^{-1}[a_{j'}][A_{j',j}].$$

Now, the matrix $[b_{(j+1),j}]$ is a function of ζ and the physical parameters of the $(j+1)^{th}$ and the j^{th} layers. We may apply eq. (4.30a) successively to relate the coefficients of any two layers. We note that eq. (4.30) or (4.30a) are recurrence relations. That is, the coefficient functions of one layer are given in terms of those of the next lower layer and the physical parameters in both layers.

A successive application of eq. (4.30a) will relate the coefficients of the first layer (j=1) to those of the n^{th} or last layer. We write a product form as follows:

$$\vec{A}_1 = \left(\prod_{\ell=1}^{n-1} [b_{(\ell+1),\ell}] \right) \vec{A}_n \ , \qquad (4.31)$$

where we may set

$$[M] = \prod_{\ell=1}^{n-1} [b_{(\ell+1),\ell}].$$

Eq. (4.31) may then be written

$$\vec{A}_1 = [M]\ \vec{A}_n \ , \qquad (4.31a)$$

where recurrence relation (4.30a) has been used to successively eliminate the coefficients of the intermediate layers. The effects of the intermediate layers are included in the [M] matrix, which is a function of ζ and the physical parameters of all the layers. One may denote the elements of the [M] matrix as m_{ij}, i, j = 1, 2, 3, 4. Using eq. (4.28a), we see we may write (4.31a) in expanded form as

$$\frac{A_1}{a_{\alpha 1}} = m_{12}\frac{B_n}{a_{\alpha n}} + m_{14}\frac{D_n}{a_{\beta n}}$$

$$\frac{B_1}{a_{\alpha 1}} = m_{22}\frac{B_n}{a_{\alpha n}} + m_{24}\frac{D_n}{a_{\beta n}} \tag{4.31b}$$

$$\frac{C_1}{a_{\beta 1}} = m_{32}\frac{B_n}{a_{\alpha n}} + m_{34}\frac{D_n}{a_{\beta n}}$$

$$\frac{D_1}{a_{\beta 1}} = m_{42}\frac{B_n}{a_{\alpha n}} + m_{44}\frac{D_n}{a_{\beta n}} \quad .$$

Equation (4.31b) is a condensed expression for the dynamics of the entire layered viscoelastic halfspace. The successive application of the recurrence relation (4.30a) given in eq. (4.31) eliminates the coefficients A_j, B_j, C_j, D_j for j = 2, 3, ..., (n-1); i.e., the explicit calculation of the potentials $\phi_{\alpha j}$ and $\phi_{\beta j}$ for the intermediate layers is not necessary. We note that the four coefficients of the first layer are related to only two coefficients in the last layer.

D. The Liquid-Solid Interaction

The dynamic field for the system (Figure 2) may be determined now using the Green's function for the liquid (4.21) and the result for the layered viscoelastic bottom [eq. (4.31b)]. The interaction of the liquid and the solid fields is determined by the boundary conditions applicable at the interface separating the two media. The boundary conditions are:

$$\begin{aligned} &\text{i)} \quad \sigma_{zz_0} = \sigma_{zz_1} = -p_0' \text{ at } z = 0 \\ &\text{ii)} \quad u_{z_0} = u_{z_1} \qquad\qquad \text{"} \\ &\text{iii)} \quad \sigma_{rz_1} = 0 \qquad\qquad\quad \text{"} \quad , \end{aligned} \tag{4.32}$$

where the subscript 0 refers to the liquid and 1 to the first viscoelastic layer.

The first two conditions are continuity of normal stress and displacement at the interface. These arise from the equation of motion and the continuity equation. The third is a consequence of continuity of stress at the interface. We note that the inviscid liquid cannot sustain a shear stress. The continuity of tangential displacement has been relaxed due to the presence of the liquid.

One evaluates eqs. (4.32) using eqs. (4.21), (4.22) and (4.24) to obtain the following matrix relation:

$$\begin{bmatrix} H(\omega)a_0\cosh a_0h_0 & -a_{\alpha 1} & a_{\alpha 1} & \zeta^2 & \zeta^2 \\ H(\omega)\rho_0\omega^2\sinh a_0h_0 & \rho_1\beta_1^2(2\zeta^2-k_{\beta 1}^2) & \rho_1\beta_1^2(2\zeta^2-k_{\beta 1}^2) & -2\rho_1\beta_1^2\zeta^2 a_{\beta 1} & 2\rho_1\beta_1^2\zeta^2 a_{\beta 1} \\ 0 & -2a_{\alpha 1} & 2a_{\alpha 1} & (2\zeta^2-k_{\beta 1}^2) & (2\zeta^2-k_{\beta 1}^2) \end{bmatrix} \times$$

$$\times \begin{bmatrix} \frac{A_0}{a_0} \\ \frac{A_1}{a_{\alpha 1}} \\ \frac{B_1}{a_{\alpha 1}} \\ \frac{C_1}{a_{\beta 1}} \\ \frac{D_1}{a_{\beta 1}} \end{bmatrix} = \frac{2H(\omega)}{a_0} \begin{bmatrix} a_0\cosh a_0z' \\ \rho_0\omega^2\sinh a_0z' \\ 0 \end{bmatrix} . \qquad (4.33)$$

Equation (4.33) has five unknown functions A_0, A_1, etc. and only three independent equations. We may reduce the number of unknown functions to three by applying the result obtained from the recurrence relation [eq. (4.31b)]. Equation (4.31b) relates the four unknown functions in the first layer to the two functions in the last layer. Applying this to eq. (4.33) gives

$$\begin{bmatrix} a_0\cosh a_0h_0 & b_{12} & b_{13} \\ \rho_0\omega^2\sinh a_0h_0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} A_0/a_0 \\ B_n/a_{\alpha n} \\ D_n/a_{\beta n} \end{bmatrix} = \frac{2}{a_0} \begin{bmatrix} a_0\cosh a_0z' \\ \rho_0\omega^2\sinh a_0z' \\ 0 \end{bmatrix} , \qquad (4.34)$$

where

$$b_{12} = a_{\alpha 1}(m_{22} - m_{12}) + \zeta^2(m_{32} + m_{42}),$$

$$b_{13} = a_{\alpha 1}(m_{24} - m_{14}) + \zeta^2(m_{34} + m_{44}),$$

$$b_{22} = \rho_1\beta_1^2(2\zeta^2 - k_{\beta 1}^2)(m_{12} + m_{22}) + (m_{42} - m_{32})2\rho_1\beta_1^2\zeta^2 a_{\beta 1},$$

$$b_{23} = \rho_1\beta_1^2(2\zeta^2 - k_{\beta 1}^2)(m_{14} + m_{24}) + (m_{44} - m_{34})2\rho_1\beta_1^2\zeta^2 a_{\beta 1},$$

$$b_{32} = 2a_{\alpha 1}(m_{22} - m_{12}) + (2\zeta^2 - k_{\beta 1}^2)(m_{32} + m_{42})$$

and

$$b_{33} = 2a_{\alpha 1}(m_{24} - m_{14}) + (2\zeta^2 - k_{\beta 1}^2)(m_{34} + m_{44}).$$

Since we are primarily interested in the acoustic field, we wish to solve for A_0 in eq. (4.34). Applying Cramer's rule gives

$$A_0 = 2\,\frac{\Delta_1}{\Delta_0} \quad , \tag{4.35}$$

where

$$\Delta_1 = K_1 a_0 \cosh(a_0 z') - K_2\rho_0\omega^2 \sinh(a_0 z')$$

and

$$\Delta_0 = K_1 a_0 \cosh(a_0 h_0) - K_2\rho_0\omega^2 \sinh(a_0 h_0).$$

The factors K_1 and K_2 are given as

$$K_1 = b_{22}b_{33} - b_{32}b_{23}$$

and

$$K_2 = b_{12}b_{33} - b_{13}b_{32}. \tag{4.35a}$$

The expression for A_0 in eq. (4.35) is applied to the Green's function expressions (4.21) to yield

$$\underset{\sim}{G}_{>} = \frac{2H(\omega)}{4\pi a_0} \sinh[a_0(h_0 - z)]\left[\frac{K_1 a_0 \cosh\, a_0 z' - K_2\rho_0\omega^2 \sinh\, a_0 z'}{K_1 a_0 \cosh\, a_0 h_0 - K_2\rho_0\omega^2 \sinh\, a_0 h_0}\right]$$

and

$$\underset{\sim}{G}_{<} = \frac{2H(\omega)}{4\pi a_0} \sinh[a_0(h_0 - z')]\left[\frac{K_1 a_0 \cosh\, a_0 z - K_2\rho_0\omega^2 \sinh\, a_0 z}{K_1 a_0 \cosh\, a_0 h_0 - K_2\rho_0\omega^2 \sinh\, a_0 h_0}\right] \tag{4.36}$$

We may apply the notation used in eq. (4.16) for the unbounded fluid to combine these two expressions for the Green's function, giving

$$\underline{G}(\zeta,z,z') = \frac{2H(\omega)}{4\pi a_0}\,[\sinh a_0(h_0-z_>)]\left[\frac{K_1a_0\cosh(a_0z_<) - K_2\rho_0\omega^2\sinh(a_0z_<)}{K_1a_0\cosh(a_0h_0) - K_2\rho_0\omega^2\sinh(a_0h_0)}\right] \qquad (4.37)$$

where, as before

$$z_> = \text{Max }(z,z')$$

and

$z_< = \min\,(z,z')$. This result is a consequence of the principle of reciprocity whereby the form of the Green's function is invariant with respect to an interchange of the source and field point. Equation (4.37) is the transformed expression for the solution to the multi-layer problem. The factors K_1 and K_2 contain all the effects of the layered viscoelastic subbottom due to their dependence on the [M] matrix, as can be seen from eqs. (4.35a) and (4.34). The actual Green's function must be obtained by taking the inverse Fourier-Bessel transform of eq. (4.37). The expression for the Green's function in the frequency domain may be written using eq. (4.4) as

$$G(r,z,z',\omega) = \int_0^\infty \underline{G}(\zeta,z,z')J_0(\zeta r)\zeta d\zeta. \qquad (4.38)$$

One may observe that the singularities of eq. (4.37) are important when performing the integration indicated in eq. (4.38). Pole singularities occur when the denominator of eq. (4.37) goes to zero, or

$$\Delta_0 = K_1a_0\cosh(a_0h_0) - K_2\rho_0\omega^2\sinh(a_0h_0) = 0. \qquad (4.39)$$

One may manipulate this into the following form

$$\tanh(a_0h_0) = \frac{K_1a_0}{K_2\rho_0\omega^2}\ ,$$

where the dependence on the water depth appears on the left-hand side and the subbottom effects are on the right.

The Green's function expressions given in eqs. (4.37) and (4.38) are too complex algebraically to analyze directly for the general case due to the dependence of these expressions on the recurrence relation. However, these forms, as developed here, are ideal for computer analysis due to the introduction of the recurrence relation [eq. (4.30a)] and its result [eq. (4.31)]. The recurrence relation reduces all calculations to (4×4) matrix operations, which can be performed easily on a computer. The general n-layer problem, if solved without benefit of a recurrence relation, would require inversion of a $(4n-2)$ square matrix. The order $(4n-2)$ of the matrix is governed by the number of coefficient functions A_j, B_j, C_j and D_j in the expression (4.22) for the potential functions for each layer. The computation time of the analysis will become excessive with a large number of layers. Applying the recurrence relation reduces the computer time for n large due to the cascading feature seen in eq. (4.31). That is, doubling the number of layers will result in an approximate doubling of computer time. Another obvious advantage of the recurrence relation is the conciseness of the notation and its generality, both of which are advantageous for computer work.

E. Special Cases

The general expression (4.37) for the Green's function for the response in a multilayered halfspace is too complex to analyze further. We now examine some special cases that are of interest. The first case we examine is that for one viscoelastic layer $(n=1)$. In this case the solid subbottom is taken as a homogeneous halfspace. This problem has been analyzed by Press and Ewing [40] previously for an elastic solid subbottom.

1) <u>One Viscoelastic Layer</u> (n=1) In this case, the recurrence relation (4.31) is not required. The [M] matrix (4.31a) and (4.31b) reduces to

$$m_{22} = 1$$
$$m_{44} = 1 \ ,$$

and the other elements are set to zero. The b_{ij} elements appearing in (4.34) reduce to the following

$$b_{12} = a_{\alpha 1}$$
$$b_{13} = \zeta^2$$
$$b_{22} = \rho_1\beta_1^2(2\zeta^2 - k_{\beta 1}^2)$$
$$b_{23} = 2\rho_1\beta_1^2\zeta^2 a_{\beta 1}$$
$$b_{32} = 2a_{\alpha 1}$$
$$b_{33} = (2\zeta^2 - k_{\beta 1}^2).$$

Applying these to the expression for K_1 and K_2 (4.35a) gives

$$K_1 = \rho_1\beta_1^2(2\zeta^2 - k_{\beta 1}^2)^2 - 4\rho_1\beta_1^2\zeta^2 a_{\alpha 1}a_{\beta 1}$$

and $K_2 = -k_{\beta 1}^2 a_{\alpha 1}$. Applying these results to the general expression for the Green's function (4.37) gives, after some rearranging

$$\underline{G}(\zeta,z,z') =$$
$$= \frac{2\Pi(\omega)}{4\pi a_0}\sinh[a_0(h_0 - z_>)]\left\{\frac{\frac{\rho_1}{\rho_0}a_0[(2\zeta^2-k_{\beta 1}^2)^2-4a_{\alpha 1}a_{\beta 1}\zeta^2]\cosh(a_0 z_<)+a_{\alpha 1}k_{\beta 1}^4\sinh(a_0 z_<)}{\frac{\rho_1}{\rho_0}a_0[(2\zeta^2-k_{\beta 1}^2)^2-4a_{\alpha 1}a_{\beta 1}\zeta^2]\cosh(a_0 h_0)+a_{\alpha 1}k_{\beta 1}^4\sinh(a_0 h_0)}\right\}. \tag{4.40}$$

This result agrees with Press and Ewing's [40] equations (26) and (27), after changing coordinate systems and notation. Our result contains a

$1/4\pi$ factor due to the Green's function formalism. We note that the introduction of viscoelasticity does not change the Green's function form. The parameters $k_{\beta 1}$ and $k_{\alpha 1}$ in eq. (4.40) become complex quantities, as do $a_{\alpha 1}$ and $a_{\beta 1}$.

Another case of interest that can be derived from the general result is the infinite-depth case. Here we simply take $h_0 \to \infty$ in eq. (4.37).

2) Infinite Liquid Layer Depth ($h_0 \to \infty$). The general expression for the Green's function reduces to the following:

$$\underline{G}(\zeta,z,z') = \frac{2H(\omega)}{4\pi a_0} e^{-(a_0 z_>)} \left[\frac{K_1 a_0 \cosh(a_0 z_<) - K_2 \rho_0 \omega^2 \sinh(a_0 z_<)}{(K_1 a_0 - K_2 \rho_0 \omega^2)}\right] . \tag{4.41}$$

Note that the frequency equation reduces to

$$K_1 a_0 - K_2 \rho_0 \omega^2 = 0.$$

If one combines the two special cases by setting n=1 and $h_0 \to \infty$, the Green's function takes the form

$$\underline{G}(\zeta,z,z') = \frac{2H(\omega)}{4\pi a_0} e^{-(a_0 z_>)} \left\{\frac{\frac{\rho_1}{\rho_0} a_0[(2\zeta^2-k_{\beta 1}^2)^2-4a_{\alpha 1}a_{\beta 1}\zeta^2]\cosh(a_0 z_<)+k_{\beta 1}^4 a_{\alpha 1}\sinh(a_0 z_<)}{\frac{\rho_1}{\rho_0} a_0[(2\zeta^2-k_{\beta 1}^2)^2 - 4a_{\alpha 1}a_{\beta 1}\zeta^2] + k_{\beta 1}^4 a_{\alpha 1}}\right\} . \tag{4.42}$$

Equation (4.42) represents the Green's function for the semi-infinite liquid over a homogeneous viscoelastic halfspace.

The result for the infinite liquid layer depth (4.41) may be rearranged and put into the following form by expanding the sinh and cosh terms in the numerator:

$$\underline{G}(\zeta,z,z') = \frac{H(\omega)}{4\pi a_0}\left[e^{-a_0(z_>-z_<)} + e^{-a_0(z_>+z_<)}\left(\frac{K_1a_0+K_2\rho_0\omega^2}{K_1a_0-K_2\rho_0\omega^2}\right)\right]. \qquad (4.43)$$

One sees that the first term is identical to eq. (4.16) for the unbounded fluid. This implies that the term represents the direct wave (through the water) from source to receiver. The second term is the response due to the presence of the viscoelastic subbottom. The second term includes the effects of reflection, refraction, surface waves etc. as will be shown in Chapter VI.

One may write eq. (4.43) for one solid layer (n=1) as follows:

$$G(\zeta,z,z') = \frac{H(\omega)}{4\pi a_0}\left[e^{-a_0(z_>-z_<)} + e^{-a_0(z_>+z_<)}\frac{N(\zeta)}{D(\zeta)}\right], \qquad (4.44)$$

where

$$N(\zeta) = ma_0\left[\left(\frac{2\zeta^2}{k_{\beta1}^2} - 1\right)^2 - \frac{4\zeta^2 a_{\alpha1}a_{\beta1}}{k_{\beta1}^4}\right] - a_{\alpha1},$$

$$D(\zeta) = ma_0\left[\left(\frac{2\zeta^2}{k_{\beta1}^2} - 1\right)^2 - \frac{4\zeta^2 a_{\alpha1}a_{\beta1}}{k_{\beta1}^4}\right] + a_{\alpha1}$$

and $\qquad m = \rho_1/\rho_0.$

Performing the inverse Fourier-Bessel transformation on the second term of eq. (4.44) will be the purpose of Chapter VI. The next chapter will be concerned with the inverse transformation of eq. (4.40).

V. GREEN'S FUNCTION FOR THE RESPONSE IN THE LIQUID LAYER OVERLYING A HOMOGENEOUS VISCOELASTIC HALFSPACE

A. The Integral Form for the Green's Function. One may write, from the result of the preceding chapter, the Green's function for n=1, or one viscoelastic layer (a halfspace)

$$\underline{G}(\zeta,z,z',\omega) = \frac{2H(\omega)}{4\pi a_0} \sinh[a_0(h_0-z_>)]\left[\frac{N(\zeta^2)}{D(\zeta^2)}\right], \tag{4.40}$$

where

$$N(\zeta^2) = ma_0[(2\zeta^2-k_\beta^2)^2 - 4a_\alpha a_\beta\zeta^2]\cosh(a_0 z_<) + a_\alpha k_\beta^4 \sinh(a_0 z_<),$$

$$D(\zeta^2) = ma_0[(2\zeta^2-k_\beta^2)^2 - 4a_\alpha a_\beta\zeta^2]\cosh(a_0 h_0) + a_\alpha k_\beta^4 \sinh(a_0 h_0),$$

$$a_{\alpha 1} = a_\alpha,\ a_{\beta 1} = a_\beta,\ k_{\beta 1} = k_\beta \text{ and } k_{\alpha 1} = k_\alpha.$$

Equation (4.40) is the Fourier-Bessel transformed form for the Green's function. We wish to perform the inverse transformation using eq. (4.38). The desired form for the Green's function in the frequency domain is written as an improper integral in the form

$$G(r,z,z',\omega) = \int_0^\infty \underline{G}(\zeta,z,z',\omega)J_0(\zeta r)\zeta d\zeta \tag{4.38}$$

or, from eq. (4.40):

$$G(r,z,z',\omega) = \frac{2H(\omega)}{4\pi}\int_0^\infty \frac{\sinh[a_0(h_0-z_>)]}{a_0}\ \frac{N(\zeta^2)}{D(\zeta^2)}\ J_0(\zeta r)\zeta d\zeta. \tag{5.1}$$

This expression is an improper integral due to the infinite upper limit and presence of singularities in the integrand. The integration of eq. (5.1) has been discussed by Press and Ewing [40] for the elastic solid case. The discussion here treats the more general case where damping or viscoelasticity is included. As discussed before, viscoelasticity effects are manifested by the imaginary component of the wavenumbers k_α and k_β in the subbottom.

We choose to manipulate eq. (5.1) so that the range of integration includes the whole line, or

$-\infty < \zeta < \infty$. To do this, we note the following relations involving zeroth order Bessel functions [28]:

$$H_0^{(1)}(x) = \frac{-2i}{\pi} \int_0^\infty e^{ix \cosh u}\, du,$$

$$H_0^{(2)}(x) = \frac{2i}{\pi} \int_0^\infty e^{-ix \cosh u}\, du \tag{5.2}$$

and $$J_0(x) = \frac{1}{2} [H_0^{(1)}(x) + H_0^{(2)}(x)].$$

The last identity is analogous to the breakdown of trigonometric functions into exponentials, as pointed out by Sommerfeld [47]. In particular, one may write

$\cos x = \frac{1}{2} [e^{ix} + e^{-ix}]$, a form directly analogous to the expression for $J_0(x)$ in eq. (5.2). The functions $J_0(x)$ and $\cos x$ may be considered standing waves, while the exponential forms and the Hankel functions $H_0^{(1)}$ and $H_0^{(2)}$ represent traveling waves. The radiating waves may be incoming or outgoing (radiating) depending on the form taken for the time dependence. The time dependence is governed by the kernel of the Fourier transform in time. We see that the kernel for the function $f(t)$ in eq. (3.9b) is $e^{i\omega t}$. Therefore, outgoing waves must have as negative exponent; e.g., they must be of the form e^{-ix}. One sees from eq. (5.2) that the integral form for $H_0^{(2)}$ is, in effect, a superposition of outgoing waves. For this reason we wish to express eq. (5.1) in terms of the Hankel function $H_0^{(2)}$. Since the argument of the Bessel function is ζr, one sees that the choice of $H_0^{(2)}(\zeta r)$ represents radiation in the r-direction. The variable of integration ζ is then a wavenumber, so the integration is over the wavenumber domain.

One also notes the following identity:

$$H_0^{(1)}(-x) = -H_0^{(2)}(x), \tag{5.3}$$

which can be verified by inspection from the integral forms for the Hankel functions in eq. (5.2). We substitute the last identity in eq. (5.2) into eq. (4.40) to obtain two terms:

$$G(r,z,z',\omega) = \frac{H(\omega)}{4\pi}\left[\int_0^{\infty} \frac{\sinh\ a_0(h_0-z_>)}{a_0} \left[\frac{N}{D}\ (\zeta^2)\right] H_0^{(2)}(\zeta r)\zeta d\zeta \ + \right.$$

$$\left. + \int_0^{\infty} \frac{\sinh\ a_0(h_0-z_>)}{a_0} \left[\frac{N}{D}\ (\zeta^2)\right] H_0^{(1)}(\zeta r)\zeta d\zeta\right] \tag{5.4}$$

The first integral is in the desired form. We change variables in the second as follows:

$$\zeta' = -\zeta$$

and $d\zeta' = -d\zeta$, where we note that $a_0(\zeta) = a_0(\zeta')$, etc. The identity (5.3) is applied to the second integral. After elementary manipulation, we write the integral form for the Green's function as:

$$G(r,z,z',\omega) = \frac{H(\omega)}{4\pi}\int_{-\infty}^{\infty} \frac{\sinh\ a_0(h_0-z_>)}{a_0} \left[\frac{N}{D}\ (\zeta^2)\right] H_0^{(2)}(\zeta r)\zeta d\zeta. \tag{5.5}$$

This form will be more convenient for evaluation using complex variable techniques and contour integration.

Equation (5.5) is now expressed in nondimensional form. We recall that ζ is a wavenumber, so it is natural to define a nondimensional variable of integration x in terms of the wavenumber k_0 in the liquid

$$x = \zeta/k_0$$

and (5.6)

$$dx = d\zeta/k_0$$

In addition, one may introduce the following nondimensional parameters:

$$\alpha = k_\alpha/k_0,$$
$$\beta = k_\beta/k_0 \tag{5.7}$$
and $$\gamma_r = k_0 r.$$

Here α and β are complex quantities if damping is present in the sub-bottom. The nondimensional parameter γ_r is a ratio of two length scales. This may be seen from the relation between the wavelength λ_0 in the liquid and the corresponding wavenumber:

$$\lambda_0 = \frac{2\pi}{k_0} = \frac{2\pi c_0}{\omega} \ , \tag{5.7a}$$

where ω is the frequency. If one applies this to the expression for γ_r in eq. (5.7), we have

$$\gamma_r = \frac{2\pi r}{\lambda_0} \text{ , a ratio of two length scales.}$$

Referring to Table 1 in Chapter I, we see that for marine sediments in shallow water

$$\mathrm{Re}\{\alpha\} < 1$$
and $$\mathrm{Re}\{\beta\} > 1.$$

The quantities a_0, a_α and a_β may be written

$$a_0 = k_0(x^2-1)^{\frac{1}{2}},$$
$$a_\alpha = k_0(x^2-\alpha^2)^{\frac{1}{2}} \tag{5.7b}$$
and $$a_\beta = k_0(x^2-\beta^2)^{\frac{1}{2}}.$$

We express eq. (5.5) in nondimensional form using eqs. (5.6) and (5.7):

$$G(\gamma_r, z_>, z_<, \omega) = \frac{k_0 H(\omega)}{4\pi}\int_{-\infty}^{\infty} \frac{\sinh[k_0(h_0-z_>)(x^2-1)^{\frac{1}{2}}]}{(x^2-1)^{\frac{1}{2}}} \left[\frac{N(x^2)}{D(x^2)}\right] H_0^{(2)}(\gamma_r x)x\,dx, \tag{5.8}$$

where

$$N(x^2) = m(x^2-1)^{\frac{1}{2}}[(2x^2-\beta^2)^2 - 4(x^2-\alpha^2)^{\frac{1}{2}}(x^2-\beta^2)^{\frac{1}{2}}x^2]\cosh[k_0 z_<(x^2-1)^{\frac{1}{2}}] + $$
$$+ \beta^4(x^2-\alpha^2)^{\frac{1}{2}}\sinh[k_0 z_<(x^2-1)^{\frac{1}{2}}]$$

and

$$D(x^2) = m(x^2-1)^{\frac{1}{2}}[(2x^2-\beta^2)^2 - 4(x^2-\alpha^2)^{\frac{1}{2}}(x^2-\beta^2)^{\frac{1}{2}}x^2]\cosh[k_0 h_0(x^2-1)^{\frac{1}{2}}]$$
$$+ \beta^4(x^2-\alpha^2)^{\frac{1}{2}}\sinh[k_0 h_0(x^2-1)^{\frac{1}{2}}].$$

We wish to evaluate the integral (5.8) using complex variable techniques. One does this by integrating around a closed curve in the complex

$$z = x + iy \tag{5.9}$$

plane, where the contour includes the real axis and the singularities of the integrand of (5.8) in the z-plane are taken into account. Note that the complex variable z is not related to the coordinate z in Figure 2. In the following, the symbols $z_>$ and $z_<$ will be used for z and z' representing coordinates to eliminate confusion.

B. Integration in the Complex Plane

The complex variable $z = x + iy$ is introduced for x in the integrand of eq. (5.8). We write the contour integral I corresponding to eq. (5.8) as:

$$I = \oint \frac{\sinh\, k_0(h_0-z_>)(z^2-1)^{\frac{1}{2}}}{(z^2-1)^{\frac{1}{2}}} \left[\frac{N(z^2)}{D(z^2)}\right] H_0^{(2)}(\gamma_r z)z\,dz, \text{ where} \tag{5.10}$$

$N(z^2)$ and $D(z^2)$ are of the same form as in eq. (5.8). We must now choose an appropriate contour so that part of it lies along the real axis. In addition, the singularities of eq. (5.10) lying within the contour must be analyzed. Two kinds of singularities occur: poles and branch points. The branch point singularities occur due to the presence of quantities taken to the 1/2 power and the Hankel function. These make the integrand of eq. (5.10) multiple-valued unless one draws cuts from the branch points to points on the contour where the integrand vanishes. One may then specify a branch (for each singularity) making the integrand single-valued provided the contour does not cross a branch cut.

Poles occur where the integrand becomes infinite; i.e. when

$$D(z^2) = 0. \tag{5.11}$$

Equations (5.11) is the frequency equation for the system. That is, solutions (roots) of this equation represent nondimensional wavenumbers, or spatial frequencies. We note that poles apparently occur at $z = \pm 1$ due to the $(z^2-1)^{1/2}$ factor in the denominator of eq. (5.10). These are not true poles due to the sinh term in the numerator. If one expands the sinh term in a power series one may write

$$\sinh k_0(h_0-z_>)(z^2-1)^{1/2} = k_0(h_0-z_>)(z^2-1)^{1/2} + \frac{[k_0(h_0-z_>)(z^2-1)^{1/2}]^3}{3!} + \ldots .$$

A $(z^2-1)^{1/2}$ factor may be pulled out of the expansion cancelling out the corresponding factor in the denominator.

Schermann [42] investigated the roots of eq. (5.11) for the case of no damping (α and β real, in our notation). He found a finite number of real roots lying in the region

$$\alpha < x < 1.$$

In addition, he found an infinite number of complex roots. Ewing, et al [7], in a discussion of Schermann's results, conclude that the complex roots do not lie in the permissible Riemann sheet. The location of the roots is discussed in detail in Appendix B. The effect of small damping is to pull the roots slightly off the real axis into the fourth quadrant for positive roots and into the second quadrant for negative ones.

At this point we determine the Riemann sheet for the integrand of eq. (5.10). We specify one branch for each of the factors introducing branch point singularities. We start with the Hankel function $H_0^{(2)}(\gamma_r z)$. This gives us a logarithmic singularity at $z = 0$. We draw a cut from $z = 0$ down the negative imaginary axis, and specify

$$-\pi/2 < \{\mathrm{Arg}\, H_0^{(2)}(\gamma_r z)\} < 3\pi/2.$$

Six additional branch points occur when

$$a_0 = 0,$$

$$a_\alpha = 0$$

and $$a_\beta = 0.$$

From eq. (5.7b) one sees that the corresponding branch points are at

$$z = \pm 1,$$

$$z = \pm\alpha$$

and $$z = \pm\beta.$$

We draw branch cuts as shown in Figure 3. Recalling the expression for a_0, a_α and a_β from eq. (5.7b), we write in the complex plane:

$$a_0 = k_0(z^2-1)^{\frac{1}{2}},$$

$$a_\alpha = k_0(z^2-\ ^2)^{\frac{1}{2}} \qquad (5.12)$$

and $a_\beta = k_0(z^2-\beta^2)^{\frac{1}{2}}$. Appropriate branches of these quantities are taken by setting

$$-\pi/2 \leq \mathrm{Arg}\{(z^2-1)^{\frac{1}{2}}\} < \pi/2,$$

$$-\pi/2 \leq \mathrm{Arg}\{(z^2-\alpha^2)^{\frac{1}{2}}\} < \pi/2$$

and $$-\pi/2 \leq \mathrm{Arg}\{(z^2-\beta^2)^{\frac{1}{2}}\} < \pi/2.$$

This is equivalent to restricting the real part of a_0, a_α and a_β to be

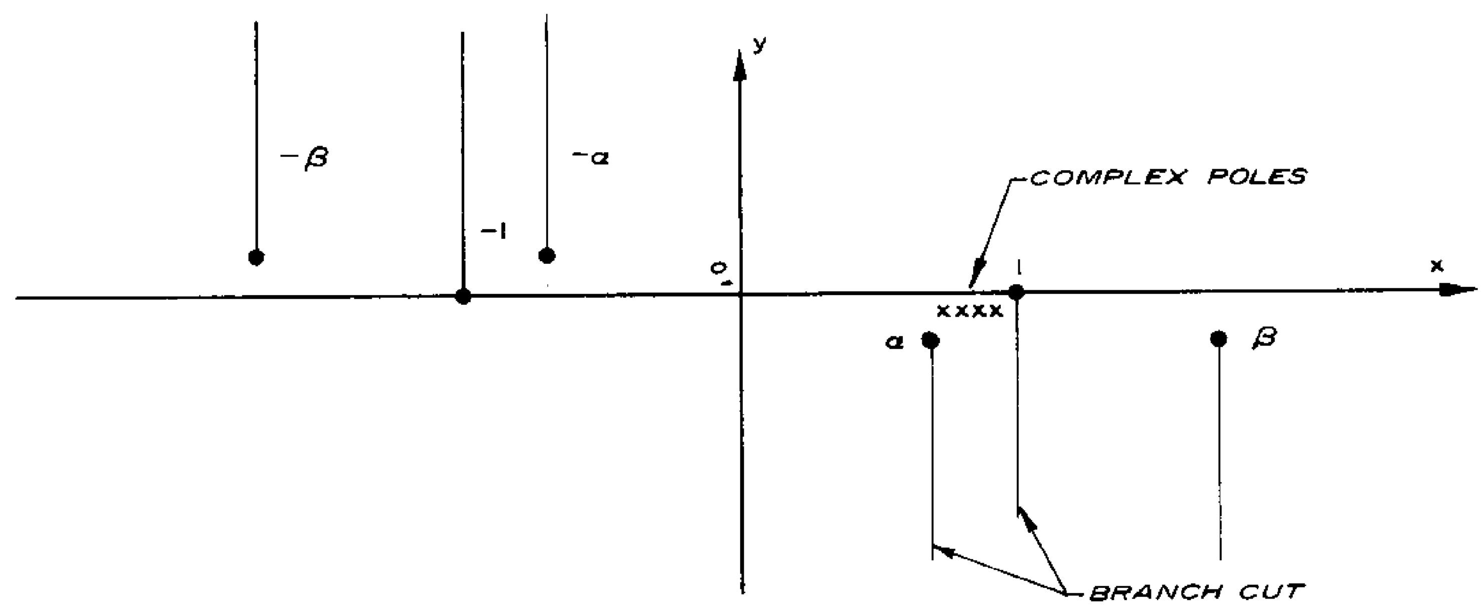


FIGURE 3

Liquid Layer Problem: Complex Plane

positive. The choice of these branches is consistent with taking the positive square root for

$$a_0(\zeta^2-k_0^2)^{\frac{1}{2}} \text{ in eq. (4.8).}$$

Having located the singularities of the integrand, we now select an appropriate contour. Part of the contour must lie on the real axis, as this part represents the Green's function eq. (5.8). We may close the contour with infinite semicircles in the upper or lower halfplanes provided the path loops around the branch cuts. To decide which semicircle to take, we recall from eq. (5.2) the integral form for the Hankel function appearing in eq. (5.10)

$$H_0^{(2)}(\gamma_r z) = \frac{2i}{\pi} \int_0^\infty e^{-\gamma_r z \cosh u} \, dn. \tag{5.2}$$

We express z along the semicircle as

$$z = Re^{i\theta} = R(\cos\theta + i \sin\theta) \tag{5.13}$$

where R is a positive real number ($R \to \infty$), and θ is the argument of z measured positive counterclockwise from the positive real axis. Substituting eq. (5.13) into (5.2) gives

$$H_0^{(2)}(\gamma_r z) = \frac{2i}{\pi} \int_0^\infty e^{-i\gamma_r R \cos\theta \cosh u} e^{\gamma_r R\sin\theta \cosh u} du.$$

The first exponential term is oscillatory, where outgoing waves (with respect to r) correspond to points in the first and fourth quadrants. This is the reason for taking the branch cuts as shown in Figure 3. That is, the cuts represent a continuous spectrum of outgoing radiation in the lateral (r) direction. The second exponential vanishes for large R in the lower halfplane due to the $\sin\theta$ term. One then must close the contour in the lower halfplance, where the integrand vanishes exponentially along the arc for $R \to \infty$. This arc will not contribute to the contour integral.

We may readily verify that the remainder of the integrand in eq. (5.10) vanishes along the lower infinite semicircle. The hyperbolic functions degenerate to exponentials for $R \to \infty$. One then writes

$$\frac{\sinh[k_0(h_0-z_>)(z^2-1)^{\frac{1}{2}}]}{(z^2-1)^{\frac{1}{2}}}\left[\frac{N(z^2)}{D(z^2)}\right] \to \frac{e^{-k_0(z_>-z_<)(z^2-1)^{\frac{1}{2}}}}{(z^2-1)^{\frac{1}{2}}}$$

as $R \to \infty$. This clearly vanishes along the semicircle. We now draw the contour as shown in Figure 4. The path of integration must loop around the branch cuts to avoid crossing them. We take the loops very close to the cuts to simplify the evaluation of the integral. The portion of the path along the real axis is deformed slightly to avoid the singularities at $z = 0$ and $z = 1$. A slight deformation of the original path along the real axis is permissible as long as there are no poles lying between the original and the deformed path. We denote the loop paths around the cuts as Γ_0, Γ_α, Γ_1 and Γ_β for the branch points at $z = 0$, α, 1 and β, respectively.

Having determined the location of the singularities and the contour, we may apply the residue theorem [3] to eq. (5.10) giving

$$I = \oint \frac{\sinh[k_0(h_0-z_>)(z^2-1)^{\frac{1}{2}}]}{(z^2-1)^{\frac{1}{2}}}\left[\frac{N(z^2)}{D(z^2)}\right] H_0^{(2)}(\gamma_r z)z\,dz =$$

$$= -\frac{4\pi}{k_0 H(\omega)} G(\gamma_r, z_>, z_<, \omega) + I_0 + I_\alpha + I_1 + I_\beta =$$

$$= 2\pi i \sum(\text{Residues}), \tag{5.14}$$

where

$$I_{0,\alpha,1,\beta} = \int_{\Gamma_{0,\alpha,1,\beta}} \frac{\sinh[k_0(h_0-z_>)(z^2-1)^{\frac{1}{2}}]}{(z^2-1)^{\frac{1}{2}}} \frac{N(z^2)}{D(z^2)} H_0^{(2)}(\gamma_r z)z\,dz,$$

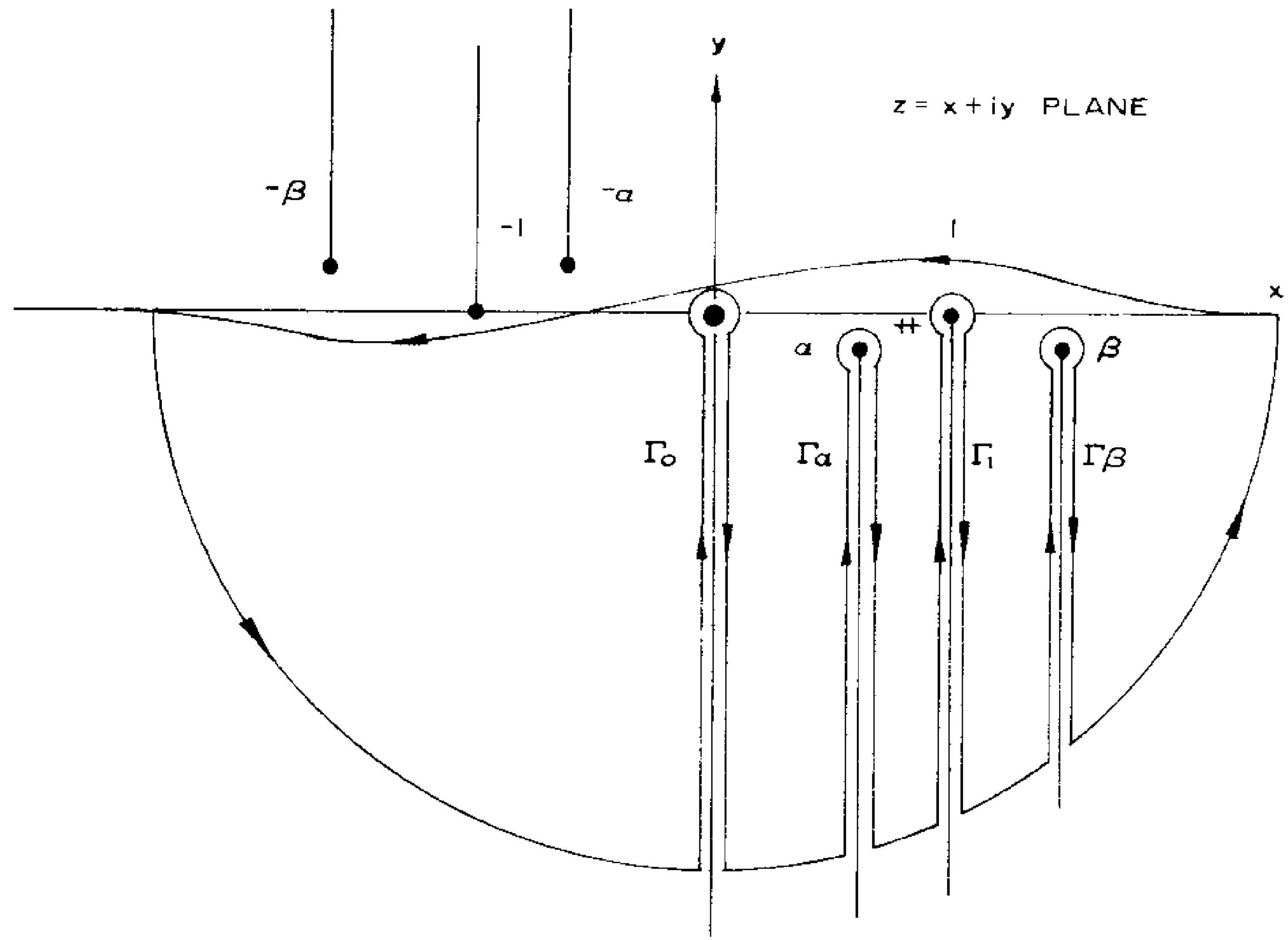


FIGURE 4

Liquid Layer Problem: Contour of Integration and the Green's function term arises from eq. (5.8). Equation (5.14) is a statement that the contour integral (consisting of the sum of the segments along the real axis and around each branch cut) is equal to the sum of its residues. We wish to find the Green's function, so after rearranging eq. (5.14) we find

$$G(\gamma_r, z_>, z_<, \omega) = \frac{k_0 H(\omega)}{4\pi} \left[I_0 + I_\alpha + I_1 + I_\beta - 2\pi i \sum(\text{Residues})\right]. \tag{5.14a}$$

Taking the contour integration replaces the improper integral along the real line [eq. (5.8)] with a sum of four loop integrals and residue terms. This may not appear to be much of a simplification, but the following discussion will show that the loop integrals, which are non-singular if the paths do not cross a pole, may be approximated by asymptotic expansions. In addition, the residue terms are shown to be an algebraic form in the frequency domain.

We start by developing the residue term in eq. (5.14a). Let us denote the i^{th} pole as z_i, where from the frequency equation

$$D(z_i^2) = 0. \tag{5.11}$$

From the residue theorem [3], one writes the i^{th} residue term by multiplying the integrand of eq. (5.10) by $(z-z_i)$ and taking a limit as $z \to z_i$, or

$$(\text{Residue})_i = \lim_{z \to z_i} (z-z_i)\left(\frac{\sinh[k_0(h_0-z_>)(z^2-1)^{\frac{1}{2}}]}{(z^2-1)^{\frac{1}{2}}} \frac{N(z^2)}{D(z^2)} H_0^{(2)}(\gamma_r z) z\right). \tag{5.15}$$

The expression (5.15) is an indeterminate form in the limit due to the $(z-z_i)$ factor in the numerator and the $D(z^2)$ term [which vanishes for $z = z_i$ from eq. (5.11)] in the denominator. We apply L'Hospital's rule, giving

$$(\text{Residue})_i = \left[\frac{\sinh[k_0(h_0-z_>)(z^2-1)^{\frac{1}{2}}]}{(z^2-1)^{\frac{1}{2}}} \quad \frac{N(z^2)}{\frac{d}{dz}[D(z^2)]} \; H_0^{(2)}(\gamma_r z)z\right]_{z=z_i} . \tag{5.15a}$$

The residue terms are then simply algebraic forms in the frequency domain. The discussion in Appendix B shows that the number of residue terms is large for the problem considered because of the high frequencies used.

The physical nature of the residue terms can be determined readily by recalling that the poles lie in the range

$$\alpha < x_i < 1,$$

where $z_i = x_i + iy_i$

and y_i is small and negative. The hyperbolic functions may be written as trigonometric functions if damping is disregarded. For example, the sinh term in eq. (5.15a) may be written approximately

$$\sinh[k_0(h_0-z_>)(x_i^2-1)^{\frac{1}{2}}] = i \sin[k_0(h_0-z_>)(1-x_i^2)^{\frac{1}{2}}] . \tag{5.16}$$

Similar expressions appear for the $N(z^2)$ and $D(z^2)$ terms.

Equation (5.16) indicates that the residue terms represent standing waves in the vertical coordinate z. The residue terms radiate laterally (in the r-direction) due to the presence of the Hankel function in eq. (5.15a). For large $\gamma_r z_i$ (far-field), the Hankel function may be written in asymptotic form [48]:

$$H_0^{(2)}(\gamma_r z_i) \cong \left[\frac{2}{\pi\gamma_r z_i}\right]^{\frac{1}{2}} e^{i\pi/4} \; e^{-i\gamma_r z_i} \tag{5.17}$$

for $\quad |\gamma_r z_i| >> 1.$

Recalling that $\gamma_r = k_0 r$, one sees that the residue term is a wave spreading with a $r^{-\frac{1}{2}}$ dependence laterally, corresponding to a two-dimensional wave. The exponential term may be written

$$e^{-i\gamma_r z_i} = e^{-ik_0 r x_i}\, e^{+k_0 r y_i}\,. \tag{5.17a}$$

The last exponential term introduces a decay or attenuation since y_i is negative. The radiative term represents waves propagating laterally at speeds varying between c_0 and c_α (for no damping in the subbottom) due to the location of the poles. One may see this by defining a phase velocity for the i^{th} mode as follows:

$$c_i = c_0/x_i\,.$$

We may then write the radiative factor in eq. (5.17a) as

$$e^{-ik_0 r x_i} = e^{-i\frac{\omega r}{c_i}}\,,$$

which shows that the wave is propagating in the r-direction at a speed c_i.

To summarize, the residue terms represent a modal or waveguide type of propagation laterally with a $r^{-\frac{1}{2}}$ spreading law. The damping in the subbottom introduces wave attenuation laterally. Each term represents a wave propagating with a distinct phase velocity c_i corresponding to a pole z_i.

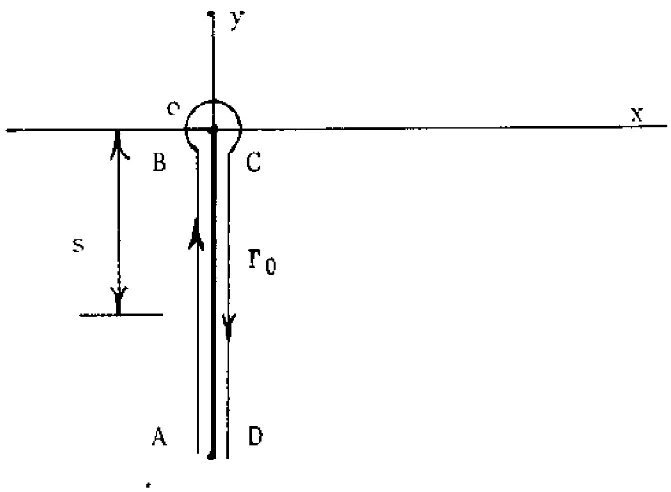
C. Integration Around the Branch Cuts.

We wish to develop expressions for the line integrals I_0, I_α, I_1 and I_β appearing in eq. (5.14). The paths $\Gamma_0, \Gamma_\alpha, \Gamma_1$ and Γ_β are indicated in Figure 4. We initiate the discussion with the integral for the branch point at z = 0.

1.) Line Integral for Path Γ_0: one writes the integral from eq. (5.14) as

$$I_0 = \int_{\Gamma_0} \frac{\sinh[k_0(h_0-z_>)(z^2-1)^{\frac{1}{2}}]}{(z^2-1)^{\frac{1}{2}}}\ \frac{N(z^2)}{D(z^2)}\, H_0^{(2)}(\gamma_r z)\,z\,dz\,, \tag{5.18}$$

where the path is indicated below.



The variable s represents the distance from the branch point. The path is essentially two straight lines along the segments AB and CD. The circular loop around $z = 0$ gives no contribution, since the point $z = 0$ is not a pole. The argument of the complex variable z increases by 2π when passing from the segment CD to AB. We write for the complex variable along CD the following

$$z = -is,$$

and along AB we write

$$z = -ise^{i2\pi} \quad . \tag{5.19}$$

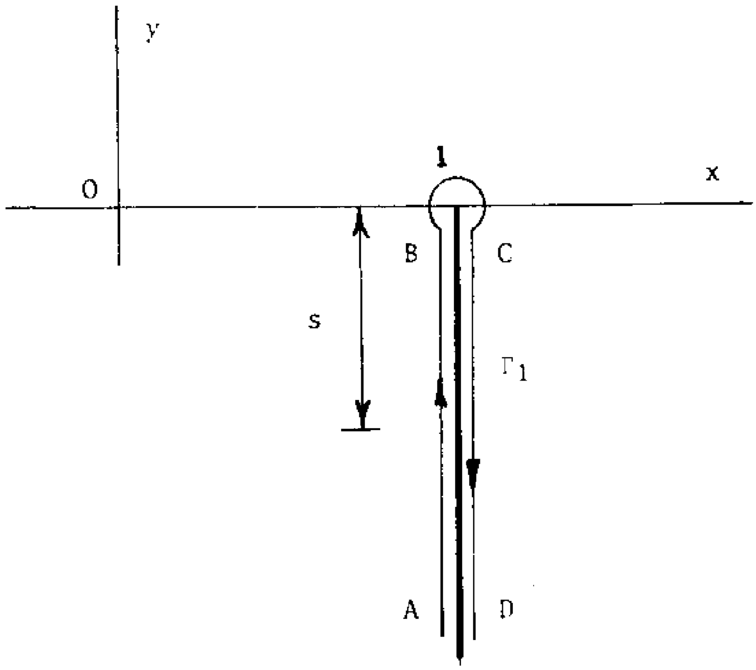
We note that $z^2 = -s^2$ on both sides of the cut, so the integrand of eq. (5.18) does not change when passing along Γ_0 from AB to CD. Integrating along s gives us symbolically

$$I_0 = \int_{AB} (\ )ds + \int_{CD} (\ )ds$$

$$= \int_{\infty}^{0} (\ )ds + \int_{0}^{\infty} (\ )ds = 0 \quad . \tag{5.20}$$

The contributions from AB and CD then cancel, resulting in $I_0 = 0$.

2.) Path Γ_1: The path Γ_1 for the branch point at $z = 1$ is shown below.



We again have two line integrals AB and CD. The circular portion gives no contribution, as $z = 1$ is not a pole. One changes variables along the two segments of the path, writing for the portion CD:

$$z = 1-is$$

and along AB: (5.21)

$$z = 1-ise^{i2\pi}.$$

Applying the change in variables gives for the quantity $(z^2-1)^{\frac{1}{2}}$ the following:

$$(z^2-1)^{\frac{1}{2}} = f(s) = is^{\frac{1}{2}}(s+2i)^{\frac{1}{2}}$$

on CD and (5.22)

$$(z^2-1)^{\frac{1}{2}} = -f(s) = -is^{\frac{1}{2}}(s+2i)^{\frac{1}{2}},$$

where the positive square root is taken on $s^{\frac{1}{2}}$. The quantity $(z^2-1)^{\frac{1}{2}}$ then changes sign from one side of the cut to the other. This results in a possible discontinuity in the integrand. One writes the integral I_1 from eq. (5.14) after introducing the function f(s) from eq. (5.22) and changing variables

$$I_1 = \int_{\infty}^{0} \frac{\sinh[-k_0(h_0-z_>)f(s)]}{-f(s)} \frac{N[s,-f(s)]}{D[s,-f(s)]} H_0^{(2)}[\gamma_r(1-is)](1-is)(-i)ds +$$

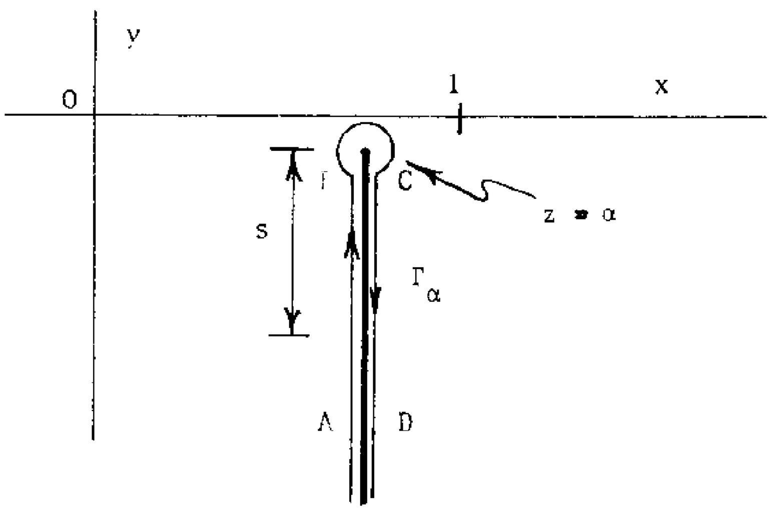
$$+ \int_0^{\infty} \frac{\sinh[k_0(h_0-z_>)f(s)]}{f(s)} \frac{N[s,f(s)]}{D[s,f(s)]} H_0^{(2)}[\gamma_r(1-is)](1-is)(-i)ds. \tag{5.23}$$

From eq. (5.8) one sees that $N[s,f(s)]$ and $D[s,f(s)]$ are even in $F(s) = (z^2-1)^{\frac{1}{2}}$. The integrands in eq. (5.23) are then identical by inspection. The two integrals cancel, giving

$$I_1 = 0. \tag{5.23a}$$

The results for the two branch points $z = 0$ and $z = 1$ indicate that these are not true singularities. We now turn to the singularity at $z = \alpha$.

3.) Path Γ_α: The path of integration is sketched as follows:



One changes variables similar to eq. (5.21). On CD we write

$$z = \alpha-is$$

and on AB

$$z = \alpha-ise^{i2\pi}. \tag{5.24}$$

The quantity $(z^2-\alpha^2)^{\frac{1}{2}}$ is discontinuous across the cut, so we write from eq. (5.24) for CD:

$$(z^2-\alpha^2)^{\frac{1}{2}} = g(s) = is^{\frac{1}{2}}(2i\alpha + s)^{\frac{1}{2}}$$

and on AB (5.25)

$$(z^2-\alpha^2)^{\frac{1}{2}} = -g(s) = -is^{\frac{1}{2}}(2i\alpha + s)^{\frac{1}{2}}.$$

Applying eqs. (5.24) and (5.25) to the expression for I_α gives, after combining the contributions along AB and CD:

$$I_\alpha = \int_0^\infty \frac{\sinh[k_0(h_0-z_>)(z^2-1)^{\frac{1}{2}}]}{(z^2-1)^{\frac{1}{2}}} H_0^{(2)}(\gamma_r z) \left[\frac{N[g(s)]}{D\,g(s)} - \frac{N[-g(s)]}{D[-g(s)]}\right] (\alpha-is)(-i)ds,$$

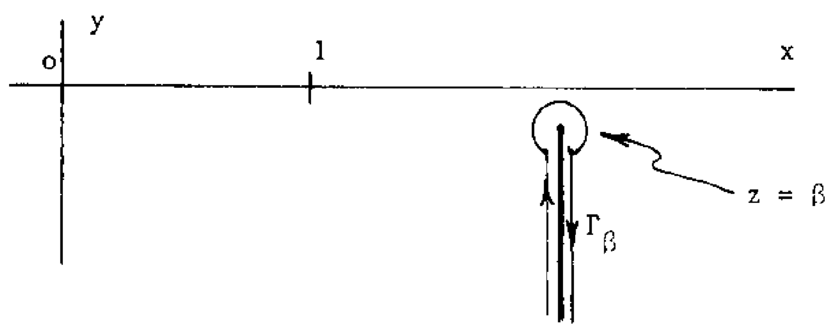
where $z = (\alpha-is)$.

The expression $(z^2-\alpha^2)^{\frac{1}{2}}$ appears only in the $N(z^2)$ and $D(z^2)$ functions. Equation (5.26) shows that only the discontinuity in the integrand due to the change of sign of $g(s)$ across the cut contributes to the integral. The quantity in brackets appearing in eq. (5.26) is odd in $g(s)$, so it may be written in the following form:

$$\left[\frac{N[g(s)]}{D[g(s)]} - \frac{N[-g(s)]}{D[-g(s)]}\right] = g(s)\ G(s), \qquad (5.26a)$$

where $G(s)$ is even in $g(s)$. The expression (5.26) for I_α is well-behaved, since the path does not intersect any poles. This may be integrated numerically, but further direct analysis cannot be applied without introducing approximations. We discuss a high-frequency far-field approximation for I_α in the following section. First we complete the branch cut integral discussion by computing I_β.

4.) Path Γ_β: We sketch the path Γ_β as follows:



The change of variables is similar in form to eq. (5.24).

$$\text{On CD:}\quad z = \beta - is.$$
$$\text{On AB:}\quad z = \beta - ise^{i2\pi}. \tag{5.27}$$

The discontinuity in the integrand now arise from the $(z^2-\beta^2)^{\frac{1}{2}}$ factor in the $N(z^2)$ and $D(z^2)$ functions. We write along CD:

$$(z^2-\beta^2)^{\frac{1}{2}} = h(s) = is^{\frac{1}{2}}(2i\beta+s)^{\frac{1}{2}}$$

and on AB (5.28)

$$(z^2-\beta^2)^{\frac{1}{2}} = -h(s) = -is^{\frac{1}{2}}(2i\beta+s)^{\frac{1}{2}}.$$

The integral I_β is similar in form to eq. (5.28)

$$I = \int_0^\infty \frac{\sinh[k_0(h_0-z_>)(z^2-1)^{\frac{1}{2}}]}{(z^2-1)^{\frac{1}{2}}} H_0^{(2)}(\gamma_r z)\left[\frac{N[h(s)]}{D[h(s)]} - \frac{N[-h(s)]}{D[-h(s)]}\right] (\beta-is)(-i)ds, \tag{5.29}$$

where $z = (\beta-is)$.

The quantity in brackets is written, similar to eq. (5.26a), as

$$\left[\frac{N[h(s)]}{D[h(s)]} - \frac{N[-h(s)]}{D[-h(s)]}\right] = h(s)H(s), \tag{5.29a}$$

where $H(s)$ is even in $h(s)$.

To summarize, two of the branch cut integrals I_0 and I_1 are zero. The other two integrals are given in eqs. (5.26) and (5.29). To analyze eqs. (5.26) and (5.29) further, we introduce a high-frequency far-field approximation. This allows us to expand the branch line integrals I_α and I_β in an asymptotic series.

D. Approximate Evaluation of the Branch Cut Integrals

In the expression for I_α [eq. (5.26)], we set

$$|\gamma_r z| >> 1,$$
$$k_0 h_0 << \gamma_r,$$
$$k_0 z_> << \gamma_r \tag{5.30}$$

and $k_0 z_< << \gamma_r$.

Equation (5.30) implies that

$$|\gamma_r \alpha| \gg 1,$$

$$h_0 \ll r,$$

$$z_> \ll r$$

and $z_< \ll r$,

since $\gamma_r = k_0 r$. These assumptions define a radiation zone in the r-direction. The first relation (5.30) allows us to express the Hankel function as follows:

$$H_0^{(2)}(\gamma_r z) \cong \left(\frac{2}{\pi k_0 r}\right)^{\frac{1}{2}} e^{i\pi/4} (\alpha - is)^{-\frac{1}{2}} e^{-ik_\alpha r} e^{-\gamma_r s}, \tag{5.31}$$

where we have used the asymptotic expression used earlier in eq. (5.17) for the Hankel function. The last three expressions in eq. (5.30) ensure that the integrand decays rapidly for increasing s due to the exponential term $e^{-\gamma_r s}$ in eq. (5.31).

We apply eqs. (5.31) and (5.26a) to eq. (5.26) to yield

$$I_\alpha = \left(\frac{2}{\pi k_0 r}\right)^{\frac{1}{2}} e^{i\pi/4} e^{-ik_\alpha r} \int_0^\infty \frac{\sinh[k_0(h_0 - z_>)(z^2-1)^{\frac{1}{2}}]}{(z^2-1)^{\frac{1}{2}}} (\alpha - is)^{\frac{1}{2}} (2i\alpha + s)^{\frac{1}{2}} s^{\frac{1}{2}} G(s) e^{-\gamma_r s} ds,$$

where $z = (\alpha - is)$. (5.32)

The integrand may be expanded about $s = 0$ as follows:

$$\frac{\sinh[k_0(h_0 - z_>)(z^2-1)]^{\frac{1}{2}}}{(z^2-1)^{\frac{1}{2}}} (\alpha - is)^{\frac{1}{2}} (2i\alpha + s)^{\frac{1}{2}} G(s) = a_0 + a_1 s + a_2 s^2 + \dots \quad . \tag{5.33}$$

This expansion is uniformly convergent inside a circle of radius ρ in the z-plane centered at $z = \alpha$. The radius ρ is given as the distance from $z = \alpha$ to the nearest singularity of the integrand of eq. (5.32).

Ignoring a remainder term, one may write I_α in expanded form using eq. (5.33) as

Recalling that $k_\alpha^R = \omega/c_\alpha$, we see that the I_α contribution to the Green's function is a damped wave propagating in the r-direction with speed c_α. We note that the contribution of I_α to the Green's function [eq. (5.14a)] is small relative to the residue terms for k_0r large due to the spreading factor $(k_0r)^{-2}$. (Recall that the residue terms spread as cylindrical waves with an $r^{-\frac{1}{2}}$ dependence.)

The expression for I_β given in eq. (5.29) is treated in a similar manner. The Hankel function is written in its asymptotic form for $z = (\beta - is)$ as follows:

$$H_0^{(2)}(\gamma_r z) \simeq \left(\frac{2}{\pi k_0 r}\right)^{\frac{1}{2}} e^{i\pi/4} (\beta - is)^{-\frac{1}{2}} e^{-ik_\beta r} e^{-\gamma_r s} . \tag{5.38}$$

We expand part of the integrand in the form

$$\frac{\sinh[k_0(h_0 - z_>)(z^2-1)^{\frac{1}{2}}]}{(z^2-1)^{\frac{1}{2}}} (\beta - is)^{\frac{1}{2}} (2i\beta + s)^{\frac{1}{2}} H(s) = b_0 + b_1 s + b_2 s^2 + \ldots \quad , \tag{5.39}$$

using eqs. (5.29a) and (5.38). The integral I_β is then expressed approximately as a descending power series in (k_0r) similar to eq. (5.37):

$$I_\beta \simeq \left(\frac{2}{\pi k_0 r}\right)^{\frac{1}{2}} e^{i\pi/4} e^{-ik_\beta r} \sum_{n=0}^{\infty} \frac{b_n \Gamma(n+3/2)}{(k_0 r)^{(n+3/2)}} . \tag{5.40}$$

This result is similar in form to the expansion for the I_α integral. The first term varies as $(k_0r)^{-2}$. The attenuation and speed of the wave have the same form. We write for k_β:

$$k_\beta = k_\beta^R + ik_\beta^I,$$

where k_β^I is real and negative. One sees that the exponential term may be written

$$e^{-ik_\beta r} = e^{-ik_\beta^R r} e^{k_\beta^I r} , \tag{5.41}$$

where the first factor on the right-hand side governs the propagation and the second the attenuation of the wave. Since one may write

$$k_\beta^R = \frac{\omega}{c_\beta} ,$$

we see that the wave propagates at the speed c_β, the shear-wave velocity of the subbottom.

E. The Approximate Green's Function

We write the simplified Green's function from eq. (5.14a) by recalling that I_0 and I_1 are zero:

$$G(\gamma_r, z_>, z_<, \omega) = \frac{k_0 H(\omega)}{4\pi} \left[I_\alpha + I_\beta - 2\pi i \; I(\text{Residues})\right]. \tag{5.42}$$

The residue terms may be written for large $|\gamma_r z_i|$ in the following form using eqs. (5.15a) and (5.17)

$$(\text{Residue})_i = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4} \, e^{-ik_0 r z_i} (k_0 r z_i)^{-\frac{1}{2}} \, c_i , \tag{5.42a}$$

where

$$c_i = - \left[\frac{\sinh k_0(h_0 - z_>)(z^2-1)^{\frac{1}{2}} \; (z^2-1)^{\frac{1}{2}} N(z^2) z}{(z^2-1)^{\frac{1}{2}} \frac{d}{dz} [D(z^2)]}\right]_{z=z_i} .$$

The branch line integrals are written from eqs. (5.37) and (5.40) as

$$I_\alpha \cong \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4} \, e^{-ik_\alpha r} \left[\frac{a_0 \Gamma(3/2)}{(k_0 r)^2} + \frac{a_1 \Gamma(5/2)}{(k_0 r)^3} + \ldots\right] \tag{5.37}$$

and

$$I_\beta \cong \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4} \, e^{-ik_\beta r} \left[\frac{b_0 \Gamma(3/2)}{(k_0 r)^2} + \frac{b_1 \Gamma(5/2)}{(k_0 r)^3} + \ldots\right] . \tag{5.40}$$

One may write the Green's function in the following form by combining eqs. (5.42a), (5.37) and (5.40) into eq. (5.42):

$$G(\gamma_r, z_>, z_<, \omega) = \frac{k_0 H(\omega)}{4\pi} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4} \left(2\pi i \sum_{i=1}^{N} \left[(k_0 r z_i)^{-\frac{1}{2}} e^{-ik_0 r z_i} c_i \right] + \right.$$

$$\left. + e^{-ik_\alpha r} \left[\frac{a_0 \Gamma(3/2)}{} + \frac{a_1 \Gamma(5/2)}{(k_0 r)^3} + \dots \right] + e^{ik_\beta r} \left[\frac{b_0 \Gamma(3/2)}{(k_0 r)^2} + \frac{b_1 \Gamma(5/2)}{(k_0 r)^{5/2}} + \dots \right] \right). \tag{5.43}$$

This result is the approximate Green's function for the high-frequency case in the far field: i.e., for horizontal distances r much larger than the other length scales in the field [see eqs. (5.17) and (5.30)].

The Green's function in the time domain may be obtained by Fourier synthesis using eq. (3.38). The synthesis is complicated, as each of the terms in eq. (5.43) is frequency-dependent. We note that the inclusion of damping makes the coefficients

$$c_i, \; i = 1, 2, \dots, N$$

$$a_0, a_1, \dots$$

$$b_0, b_1, \dots$$

frequency-dependent, as are the wavenumbers k_α and k_β. In addition, we note that the poles z_i appearing in the residue term are frequency-dependent whether damping is included or not [see Appendix B].

Pekeris analyzed the time-domain behavior of the residue terms for the two liquid layer problem [40]. His analysis is a special case of this one, but the general features of his result apply here as well. He took a pulse shape in time of the form

$$h(t) = \begin{cases} e^{-\sigma t} & t > 0 \\ 0 & t < 0, \end{cases}$$

where σ is a positive real constant. Pekeris performed the Fourier synthesis for this pulse shape using Kelvin's method of stationary phase. To the first approximation, the stationary phase treatment gives an r^{-1} dependence. Pekeris also took the next higher approximation valid near the stationary value of group velocity. This resulted in an $r^{-5/6}$ dependence in the time domain. He termed this type of wave the Airy phase.

Other types of time dependence will give similar results. In particular, a Gaussian pulse modulated by a frequency ω_0 is representative of the time dependence used in acoustic sounding. We may write h(t) in the following form:

$$h(t) = e^{-(t/\lambda)^2} e^{i\omega_0 t} , \tag{5.44}$$

where λ is a parameter related to the pulse length. The Fourier transform $H(\omega)$ of eq. (5.44) is:

$$H(\omega) = \lambda(\pi)^{\frac{1}{2}} e^{-\left[\frac{\lambda(\omega-\omega_0)}{2}\right]^2} . \tag{5.44a}$$

One may apply this pulse spectrum to eq. (5.43) and apply the stationary phase method to obtain the Green's function in the time domain in a manner similar to Pekeris' treatment.

Further development of the residue terms is beyond the scope of this investigation. The Green's function appearing in eq. (5.43) is too complex to be used for the intended application. In addition, the assumptions used to obtain this expression place us in a far field, implying that the acoustic receiver must be separated horizontally many water depths from the source. This is inconvenient for experimental work (e.g., acoustic sounding).

We terminate the finite water depth case developed here by noting that the response consists of a large finite number N of modes corresponding to terms in a residue series. These modes may interfere constructively due to the presence of large parameters in the Fourier synthesis. In addition, contributions to the response are given by the branch cut integrals. These terms die out rapidly due to their representation as a descending power series in r, the leading term being proportional to r^{-2}.

VI. GREEN'S FUNCTION FOR THE SEMI-INFINITE LIQUID OVERLYING A HOMOGENEOUS VISCOELASTIC HALFSPACE

A. Manipulation of Integral Form

We develop an expression for the Green's function for a field consisting of a liquid halfspace overlying a semi-infinite viscoelastic solid. The Fourier-Bessel transformed form of the Green's function was given as a special case of the result for the n-layered solid halfspace. The number n of layers in the solid was taken as 1 and the liquid layer depth h_0 was taken to infinity. We write the simplified result from eq. (4.44) as:

$$\underline{G}(\zeta,z_>,z_<,\omega) = \frac{H(\omega)}{4\pi a_0}\left[e^{-a_0(z_>-z_<)} + e^{-a_0(z_>+z_<)}\,\frac{N_1(\zeta^2)}{D_1(\zeta^2)}\right], \tag{4.44}$$

where

$$N_1(\zeta^2) = m\,a_0\left[\left(\frac{2\zeta^2}{k_\beta^2} - 1\right)^2 - \frac{4\zeta^2}{k_\beta^4}\,a_\alpha a_\beta\right] - a_\alpha,$$

$$D_1(\zeta^2) = m\,a_0\left[\left(\frac{2\zeta^2}{k_\beta^2} - 1\right)^2 - \frac{4\zeta^2}{k_\beta^4}\,a_\alpha a_\beta\right] + a_\alpha$$

and $m = \rho_1/\rho_0$. We have set $a_{\alpha 1} = a_\alpha$, etc. for simplicity. The first term in eq. (4.44) is the direct wave; that is, the wave traveling directly from source to receiver. This is evident, as this term is identical to the Green's function for the unbounded fluid [eq. (4.16)]. The second term is evidently the contribution due to the presence of the viscoelastic halfspace. The Green's function (4.44) may be written in the form

$$\underline{G} = \underline{G}_\infty + \underline{G}_1\ , \tag{6.1}$$

where $\underline{G}_\infty$, the direct wave, is given by eq. (4.16), and $\underline{G}_1$, the contribution due to the solid halfspace, is written as

$$\underline{G}_1 = \frac{H(\omega)}{4\pi a_0}\left[e^{-a_0(z_>+z_<)}\,\frac{N_1(\zeta^2)}{D_1(\zeta^2)}\right].$$

After noting that

$$N_1(\zeta^2) = D_1(\zeta^2) - 2a_\alpha$$

from eq. (4.44), we decompose $\underline{G}_1$ into two terms as follows:

$$\underline{G}_1 = \underline{G}_I + \underline{G}', \tag{6.2}$$

where

$$\underline{G}_I = \frac{H(\omega)}{4\pi} \frac{e^{-a_0(z_> + z_<)}}{a_0}$$

and

$$\underline{G}' = \frac{-H(\omega)}{2\pi a_0} e^{-a_0(z_> + z_<)} \frac{a_\alpha}{D_1(\zeta^2)} .$$

We note that $\underline{G}_I$ is in the same form as eq. (4.16) for the Green's function in the unbounded fluid.

To perform the inverse transform, we recall eq. (4.38):

$$G(r, z_>, z_<, \omega) = \int_0^\infty \underline{G}(\zeta, z_>, z_<, \omega) J_0(\zeta r)\zeta d\zeta.$$

Performing the inverse transformation of $\underline{G}_\infty$ gives, from eq. (4.18):

$$G_\infty = \int_0^\infty \underline{G}_\infty J_0(\zeta r)\zeta d\zeta = \frac{H(\omega)}{4\pi} \frac{e^{-ik_0 R}}{R} , \tag{4.18}$$

where $R = [(z_> - z_<)^2 + r^2]^{\frac{1}{2}}$.

One may write the Green's function G_I in a form similar to eq. (4.18). From eqs. (6.2) and (4.38) we write

$$G_I = \frac{H(\omega)}{4\pi} \int_0^\infty \frac{e^{-a_0(z_> + z_<)}}{a_0} J_0(\zeta r)\zeta d\zeta = \frac{H(\omega)}{4\pi} \frac{e^{-ik_0 R_I}}{R_I} , \tag{6.3}$$

where $R_I = [(z_> + z_<)^2 + r^2]^{\frac{1}{2}}$. The term G_I can be interpreted as an image source term using an argument similar to Sommerfeld's [49]. The image source is sketched in Figure 5.

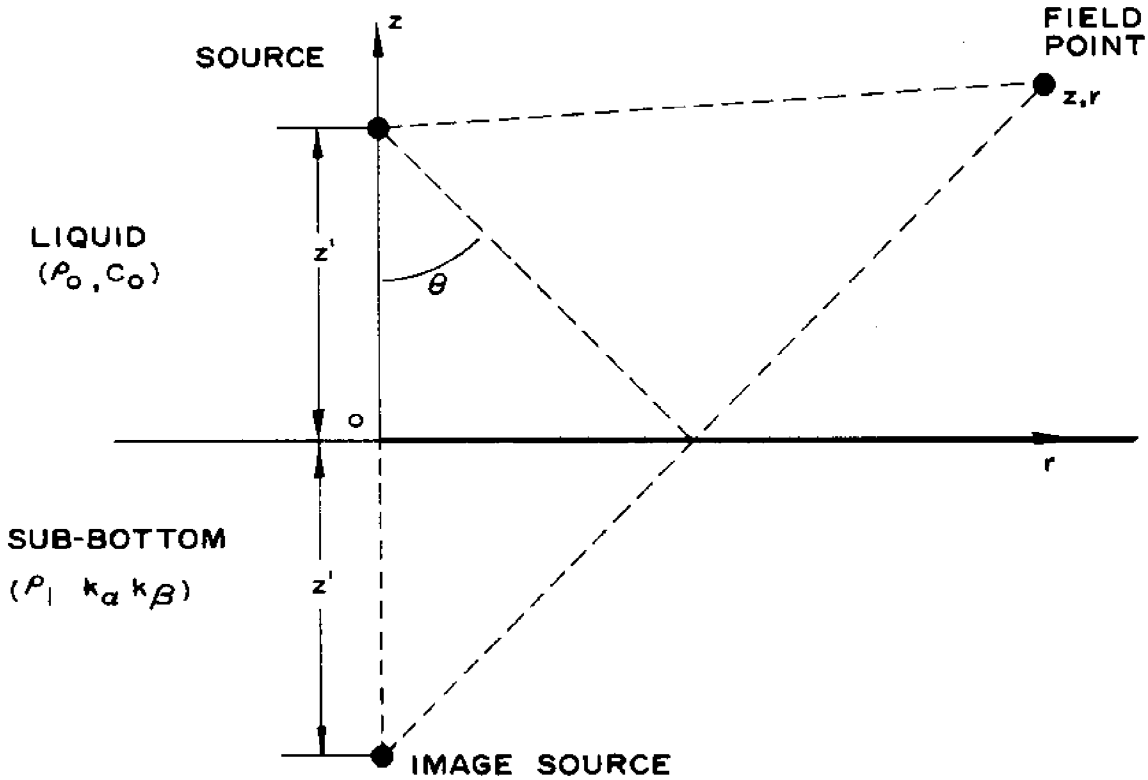


FIGURE 5

Geometry of the Liquid-Subbottom Field: Infinite Liquid Depth

The Green's function can be written as the sum of three terms: the direct wave, the image source term and a residual term. From eqs. (6.1), (6.2), (4.18) and (6.3) we write

$$G(r,z_>,z_<,\omega) = \frac{H(\omega)}{4\pi}\left[\frac{e^{-ik_0R}}{R} + \frac{e^{-ik_0R_I}}{R_I}\right] + G'(r,z_>,z_<,\omega), \tag{6.4}$$

where the residual term G' is

$$G' = \frac{-H(\omega)}{2\pi}\int_0^\infty \frac{e^{-a_0(z_>+z_<)}}{a_0}\,\frac{a_\alpha}{D_1(\zeta^2)}\,J_0(\zeta r)\zeta d\zeta.$$

Having obtained expressions for the direct and image source terms, we now direct attention to the integral expression for the residual term G' given in eq. (6.4). This integral is in the same form as the Green's function for the finite liquid layer [eq. (5.1)]. We may manipulate the integral form for G' similar to the development in Chapter V. Using eqs. (5.2), we write for the residual term:

$$G'(r,z_>,z_<,\omega) = \frac{-H(\omega)}{4\pi}\int_0^\infty \frac{e^{-a_0(z_>+z_<)}}{a_0}\,\frac{a_\alpha}{D_1(\zeta^2)}\,H_0^{(2)}(\zeta r)\zeta d\zeta. \tag{6.5}$$

This result is expressed in nondimensional form using eqs. (5.6) and (5.7). In addition, a nondimensional ratio of length scales $\gamma_z = k_0(z_>+z_<)$ is used:

$$G'(\gamma_r,\gamma_z,\omega) = \frac{-H(\omega)k_0}{4\pi}\int_{-\infty}^\infty \frac{e^{-(x^2-1)^{\frac{1}{2}}\gamma_z}}{(x^2-1)^{\frac{1}{2}}}\,\frac{(x^2-\alpha^2)^{\frac{1}{2}}}{D_1(x^2)}\,H_0^{(2)}(\gamma_r x)x dx, \tag{6.6}$$

where

$$D_1(x^2) = m(x^2-1)^{\frac{1}{2}}\left[\left(\frac{2x^2}{\beta^2}-1\right)^2 - \frac{4x^2(x^2-\alpha^2)^{\frac{1}{2}}(x^2-\beta^2)^{\frac{1}{2}}}{\beta^4}\right] + (x^2-\alpha^2)^{\frac{1}{2}}.$$

B. Contour Integration

The integral (6.6) is improper due to the presence of singularities on the x-axis. We choose to evaluate eq. (6.6) using contour integration in the complex ($z = x+iy$) plane, as was done in the preceding chapter. We write a contour integral from eq. (6.6) as follows:

$$I' = \oint \frac{e^{-(z^2-1)^{\frac{1}{2}}\gamma_z}}{(z^2-1)^{\frac{1}{2}}} \; \frac{(z^2-\alpha^2)^{\frac{1}{2}}}{D_1(z^2)} \; H_0^{(2)}(\gamma_r z)z\,dz \tag{6.7}$$

The contour of integration must be determined. The branch point singularities of the integrand in eq. (6.7) are identical to those in the preceding chapter. The poles of eq. (6.7) are given by

$$D_1(z^2) = 0 \tag{6.8}$$

Solutions to eq. (6.8) were obtained numerically by Strick and Ginsbarg for the elastic solid; i.e., for α and β real and positive [52]. They found one real root of eq. (6.8) occurring at a wavenumber x_p larger than β, or

$$\alpha < 1 < \beta < x_p. \tag{6.8a}$$

The root x_p of eq. (6.8) represents a Stoneley wave contribution [8]. The points $z = \pm 1$ are not poles. This can be seen from the Green's function given in eq. (4.42) before decomposition into the sum of two terms. Solutions to the frequency equation (6.8) with damping are discussed in Appendix C. The result for small damping is that the pole x_p is pulled off the real axis slightly into the fourth quadrant. The complex pole z_p may be written:

$$z_p = x_p + iy_p, \tag{6.8b}$$

where y_p is real and negative.

The singularities of eq. (6.7) then are similar to those in the previous chapter, except that only one pole occurs farther out the x-axis [see Figure 6]. The same contour may be used for the present problem because of the presence of the Hankel function. The integrand of eq. (6.7) vanishes along the arc of a semicircle of large radius R in the lower halfplane, as can be seen by inspection after recalling that the branches for the quantities

$$(z^2 - \alpha^2)^{\frac{1}{2}}, \ (z^2-1)^{\frac{1}{2}},$$

and $(z^2-\beta^2)^{\frac{1}{2}}$ are taken so that the real parts are positive [from eq. (5.12)].

The contour of integration is shown in Figure 6. The contour is identical to the one in the previous chapter except for the location of the poles. The branch point singularity at z=0 due to the Hankel function is ignored as it does not contribute to the integral.

Having selected a contour and knowing the location of the singularities, we apply the residue theorem to eq. (6.7), giving

$$I' = - \frac{4\pi}{k_0 H(\omega)} G'(\gamma_r,\gamma_z,\omega) + I_\alpha + I_1 + I_\beta = 2\pi i \text{ Residue}, \tag{6.9}$$

where

$$I_{\alpha,1,\beta} = \int_{\Gamma_{\alpha,1,\beta}} \frac{e^{-(z^2-1)^{\frac{1}{2}}\gamma_z}(z^2-\alpha^2)^{\frac{1}{2}}}{(z^2-1)^{\frac{1}{2}}\, D_1(z^2)} H_0^{(2)}(\gamma_r z) z dz.$$

The expressions I_α, I_1, I_β represent line integrals around the branch cuts for $z = \alpha$, 1 and β. Each line integral path is designated by $\Gamma_{\alpha,1,\beta,}$ respectively.

We may solve for the Green's function G' from eq. (6.9), giving

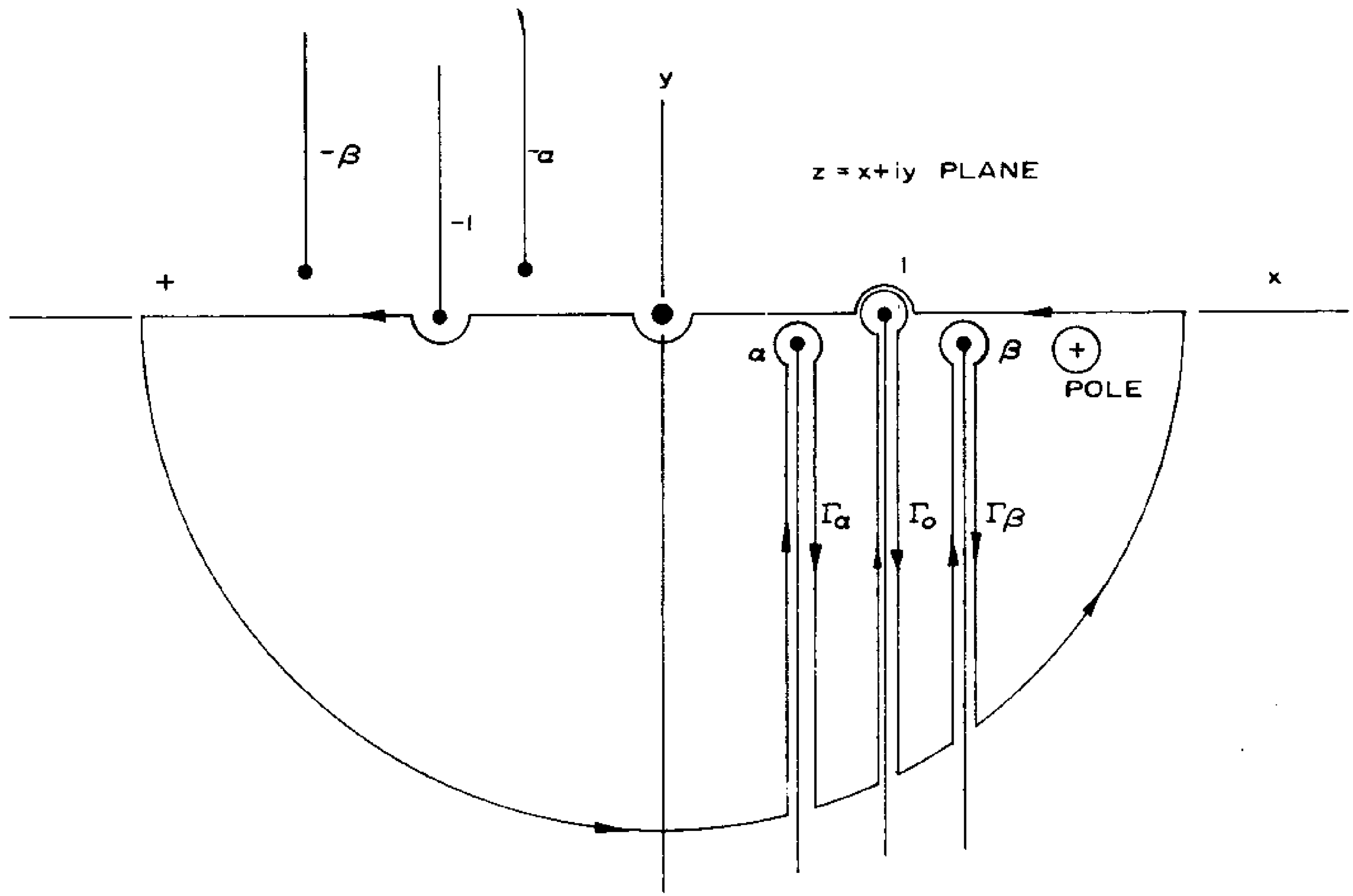


FIGURE 6

Contour of Integration: Near-Bottom Case

$$G'(\gamma_r,\gamma_z,\omega) = \frac{k_0 H(\omega)}{4\pi}\left[I_1 + I_\alpha + I_\beta - 2\pi i\,(\text{Residue})\right]. \tag{6.10}$$

The Green's function is then the sum of contributions from loops around three branch cuts and a residue term. One may write the residue term as follows:

$$(\text{Residue}) = H_0^{(2)}(\gamma_r z_p)c_p, \tag{6.11}$$

where

$$c_p = -\left[\frac{e^{-(z^2-1)^{\frac{1}{2}}z}}{\frac{d}{dz}\,[D_1(z^2)]}\,\frac{(z^2-\alpha^2)^{\frac{1}{2}}}{(z^2-1)^{\frac{1}{2}}}\,z\right]_{z=z_p}.$$

L'Hospital's rule has been applied to obtain the expression for c_p, as in Chapter V for the residues. The residue term is a wave propagating laterally due to the presence of the Hankel function. We may obtain further insight into this term by writing the Hankel function in its asymptotic form for large $|\gamma_r z_p|$. We have

$$H_0^{(2)}(\gamma_r z_p) \cong \left(\frac{2}{\pi\gamma_r z_p}\right)^{\frac{1}{2}} e^{i\pi/4}\, e^{-i\gamma_r z_p}. \tag{6.12}$$

One may apply eq. (6.8b) to the exponential term, giving

$$e^{-i\gamma_r z_p} = e^{-ik_0 r x_p}\, e^{k_0 r y_p}. \tag{6.12a}$$

The first term governs the radiative behavior of the residue term and the second is an attenuation. One may define a phase velocity c_p for the residue term as follows:

$$c_p = \frac{\omega}{k_0 x_p}. \tag{6.12b}$$

We write the radiative term as:

$$e^{-ik_0 r x_p} = e^{-\frac{\omega r}{c_p}}. \tag{6.12c}$$

Recalling that the time dependence for the wave given by the kernel of eq. (3.9b) is

$$e^{i\omega t},$$

one may combine this with eq. (6.12c) to obtain

$$e^{i\omega(t-\frac{r}{c_p})} = e^{-i\frac{\omega}{c_p}(r-c_p t)} . \qquad (6.12d)$$

This result represents a wave propagating in the r-direction with the speed c_p, vindicating calling c_p the phase velocity. Since the damping is usually small, we may further discuss the behavior of the residue term by taking

$$z_p \cong x_p .$$

In this case, the exponential term in the c_p expression (6.11) reduces to

$$e^{-(x_p^2-1)^{\frac{1}{2}}\gamma_z} = e^{-(x_p^2-1)^{\frac{1}{2}}k_0(z_>+z_<)} .$$

This represents an attenuation in the vertical direction, as $x_p > 1$. In addition, the Hankel function (6.12) introduces a $(\gamma_r x_p)^{-\frac{1}{2}} = (k_0 r x_p)^{-\frac{1}{2}}$ dependence to the residue term.

C. Branch Line Integrals

We wish to evaluate I_1, I_α and I_β given in eq. (6.9). The procedure is the same as in Chapter V, although now the integrand is less complicated algebraically. We evaluate first around the path Γ_1, changing variables using eq. (5.21). We write the quantity

$(z^2-1)^{\frac{1}{2}} = f(s)$ on the right-hand side of the cut and change its sign on the left-hand side as in eq. (5.22). We combine the two segments of the path Γ_1 into one integral in s as follows:

$$I_1 = \int_0^{\infty} \left[\frac{e^{-f(s)\gamma_z}}{D_1[f(s)]} + \frac{e^{f(s)\gamma_z}}{D_1[-f(s)]} \right] \frac{a_\alpha H_0^{(2)}[\gamma_r(1-is)]}{f(s)} (1-is)(-i)ds, \qquad (6.13)$$

where we write D_1 from eq. (6.6) in the form

$$D_1[f(s)] = mf(s)P(s) + a_\alpha(s)$$

and

$$P(s) = \left[\left(\frac{2z^2}{\beta^2} - 1\right)^2 - \frac{4z^2(z^2-\alpha^2)^{\frac{1}{2}}(z^2-\beta^2)^{\frac{1}{2}}}{\beta^4} \right]_{z=1-is}.$$

One may manipulate eq. (6.13) into the more convenient form

$$I_1 = \int_0^{\infty} [f(s)\ F_1(s) + \frac{F_2(s)}{f(s)}]\ H_0^{(2)}\ \gamma_r(1-is)\ (1-is)(-i)ds, \qquad (6.14)$$

where

$$F_1(s) = \frac{2mP\ \sinh(f\gamma_z)a}{f[a_\alpha^2 - m^2f^2P^2]}$$

and

$$F_2(s) = \frac{2a^2\ \cosh(f\gamma_z)}{[a_\alpha^2 - m^2f^2P^2]}.$$

We note that both $F_1(s)$ and $F_2(s)$ are even in $f(s)$. The integral (6.14) is exact. In the next section, the integral will be developed in an asymptotic series by applying some assumptions and approximations.

Next, we turn our attention to the integral I_α. We change variables using eq. (5.24). Recall from eq. (5.25)

$g(s) = (z^2-\alpha^2)^{\frac{1}{2}}$ on the right-hand side of the cut. The expression for I_α is written using the same methodology:

$$I_\alpha = \int_0^{\infty} \frac{e^{-(z^2-1)^{\frac{1}{2}}\gamma_z}}{(z^2-1)^{\frac{1}{2}}} H_0^{(2)}(\gamma_r z)(\alpha-is)g(s)G_1(s)(-i)ds, \qquad (6.15)$$

where

$$G_1(s) = \frac{D_1[g(s)] + D_1[-g(s)]}{D_1[g(s)]D_1[-g(s)]}$$

and $\qquad z = (\alpha-is).$

One notes that $G_1(s)$ is even in $g(s)$. The expression (6.15) is also exact. Approximate results will be developed later.

Finally, we evaluate I_β. The change in variables is given by eq. (5.27). We write from eq. (5.28)

$$(z^2-\beta^2)^{\frac{1}{2}} = h(s) = is^{\frac{1}{2}}(2i\beta+s)^{\frac{1}{2}}$$

on the right-hand side of the cut. The exact expression for I_β is written

$$I_\beta = \int_0^\infty \frac{e^{-(z^2-1)^{\frac{1}{2}}\gamma_z}}{(z^2-1)^{\frac{1}{2}}} (z^2-\alpha^2)^{\frac{1}{2}}(\beta-is)H_0^{(2)}[\gamma_r(\beta-is)]h(s)H_1(s)(-i)ds, \tag{6.16}$$

where

$$h(s)\ H_1(s) = \frac{D_1[-h(s)] - D_1[h(s)]}{D_1[h(s)]D_1[-h(s)]} \quad \text{and } z = (\beta-is).$$

We see that $H_1(s)$ is even in $h(s)$. Equation (6.16) will be evaluated approximately in the same manner as the results for I_1 and I_α.

The Green's function (6.10) has been expressed (in the frequency domain) as the sum of three definite integrals; eqs. (6.14), (6.15) and (6.16), and an algebraic residue term, eq. (6.11). The physical behavior of the residue term has been discussed. To gain insight into the definite integrals associated with the branch cuts we must evaluate the integrals approximately by introducing appropriate physical assumptions.

The most obvious assumption is to use the asymptotic expression for the Hankel function appearing in all three integrals. This requires that the magnitude of the parameters γ_r, $\alpha\gamma_r$ and $\beta\gamma_r$ be large. Taking these as large is eesentially a high-frequency assumption, as we may write for γ_r:

$$\gamma_r = k_0 r = \frac{\omega}{c_0}\ r.$$

Recalling eq. (5.7a), we see that γ_r is a ratio of r to the wavelength of a sound wave in the liquid. We note that both α and β are of the order unity, so one may say that the high-frequency assumption implies

$$\gamma_r >> 1,$$
$$|\alpha\gamma_r| >> 1 \tag{6.17}$$
and $$|\beta\gamma_r| >> 1 .$$

We begin by applying the high-frequency assumption to eq. (6.14) for I_1. We write the asymptotic form for the Hankel function as follows:

$$H_0^{(2)}[\gamma_r(1-is)] \approx (\tfrac{2}{\pi})^{\frac{1}{2}}\, \gamma_r^{-\frac{1}{2}}(1-is)^{-\frac{1}{2}}\, e^{i\pi/4}\, e^{-i\gamma_r}\, e^{-\gamma_r s} \tag{6.18}$$

The expression for I_1 becomes, for $\gamma_r >> 1$:

$$I_1 \approx (\tfrac{2}{\pi})^{\frac{1}{2}}\, \gamma_r^{-\frac{1}{2}}\, e^{i\pi/4}\, e^{-i\gamma_r}\int_0^\infty e^{-\gamma_r s}(1-is)^{+\frac{1}{2}}(-i)\left[f(s)F_1(s) + \frac{F_2(s)}{f(s)}\right]ds \tag{6.19}$$

Recalling that $f(s) = is^{\frac{1}{2}}(2i-s)^{\frac{1}{2}}$ enables us to write eq. (6.19) in the form:

$$I_1 \cong (\tfrac{2}{\pi})^{\frac{1}{2}}\gamma_r^{-\frac{1}{2}}\, e^{i\pi/4}\, e^{-i\gamma_r} \times$$
$$\times \int_0^\infty e^{-\gamma_r s}\, s^{\frac{1}{2}}(2i-s)^{\frac{1}{2}}(1-is)^{\frac{1}{2}}\, F_1(s)ds - \int_0^\infty \frac{e^{-\gamma_r s}(1-is)^{\frac{1}{2}}}{s^{\frac{1}{2}}(2i-s)^{\frac{1}{2}}}\, F_2(s)ds. \tag{6.19a}$$

One notes for $\gamma_r >> 1$, both integrands decay rapidly due to the $e^{-\gamma_r s}$ term provided $F_1(s)$ and $F_2(s)$ are properly behaved. We recall from eq. (6.14) the expressions for F_1 and F_2:

$$F_1(s) = \frac{2mP\,\sinh[f(s)\gamma_z]\; a_\alpha(s)}{f(s)[a_\alpha^2-m^2f^2P^2]}$$
and
$$F_2(s) = \frac{2a_\alpha \cosh\, f(s)\gamma_z}{[a_\alpha^2-m^2f^2P^2]} . \tag{6.14}$$

The sinh and cosh terms reduce to exponentials of the form

$\frac{1}{2} e^{\gamma_z s}$ for large s. We must set

$$\gamma_z < \gamma_r \tag{6.20}$$

for convergence of the integrals in eq. (6.19a). This second assumption may also be written as

$$(z_> + z_<) < r,$$

which, from Figure 5, implies that the horizontal range must be larger than the vertical range along the reflected path, or the angle of incidence must be less than 45^o. The convergence of the integrals in eq. (6.19a) will improve as the ratio

$\frac{r}{(z_> + z_<)} = \frac{\gamma_r}{\gamma_z}$ increases. That is, as we approach a near-bottom or low incidence condition. We may expand the integrands about s = 0 if eq. (6.15) holds. We write the appropriate expansions as follows:

$$(2i-s)^{\frac{1}{2}}(1-is)^{\frac{1}{2}}F_1(s) = b_0 + b_1 s + b_2 s^2 + \ldots$$

and

$$\frac{(1-is)^{\frac{1}{2}}}{(2i-s)^{\frac{1}{2}}} F_2(s) = c_0 + c_1 s + c_2 s^2 + \ldots \quad . \tag{6.21}$$

The expansions have a finite radius of convergence in the complex plane extending from the singularity z=1 to the nearest other singular point. The integration from s=0 to s=∞ takes us outside this radius of convergence, so term-by-term integration of eq. (6.19a) using the expansions (6.21) will give a divergent series. One takes the first few terms (at most) of the expansion, which will turn out to be an asymptotic series in descending powers of the large parameter γ_r. The first few terms usually give an accurate approximation to the integral. An upper bound to the remainder term due to the finite radius of convergence is obtained by Van Der Waerden [56].

We formally introduce the expansions (6.21) into the integral expressions giving:

$$I_1 \doteq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \gamma_r^{-\frac{1}{2}} e^{i\pi/4} e^{-i\gamma/r} \times$$

$$\times \left[\sum_{n=0}^{\infty} b_n \int_0^{\infty} s^{(n+\frac{1}{2})} e^{-\gamma_r s} ds - \sum_{n=0}^{\infty} c_n \int_0^{\infty} s^{(n-\frac{1}{2})} e^{-\gamma_r s} ds\right]. \tag{6.22}$$

The definite integrals may be expressed as gamma functions from eq. (5.35). We write

$$I_1 \doteq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4} e^{-i\gamma_r} \times$$

$$\times \left[\sum_{n=0}^{\infty} b_n \frac{\Gamma(n+3/2)}{\gamma_r^{(n+2)}} - \sum_{n=0}^{\infty} c_n \frac{\Gamma(n+1/2)}{\gamma_r^{(n+1)}}\right]. \tag{6.23}$$

Now we evaluate explicitly the first term in each series, obtaining a zeroth approximation to I_1:

$$I_{10} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4} e^{-i\gamma_r} \left[b_0 \frac{\Gamma(3/2)}{\gamma_r^2} - c_0 \frac{\Gamma(1/2)}{\gamma_r}\right]. \tag{6.24}$$

We note the values for the gamma functions of half-order [29]:

$$\Gamma(1/2) = \pi^{\frac{1}{2}}$$

and

$$\Gamma(3/2) = \frac{\pi^{\frac{1}{2}}}{2}.$$

Equation (6.24) reduces to

$$I_{10} = 2^{\frac{1}{2}} e^{i\pi/4} e^{-i\gamma_r} \left[b_0 \frac{1}{2\gamma_r^2} - c_0 \frac{1}{\gamma_r}\right]. \tag{6.24a}$$

From eq. (6.16), one writes

$$b_0 = 2^{\frac{1}{2}} i^{\frac{1}{2}} F_1(0) \tag{6.25}$$

and $$c_0 = 2^{-\frac{1}{2}} i^{-\frac{1}{2}} F_2(0).$$

From eq. (6.14), we evaluate $F_1(0)$ and $F_2(0)$ as follows:

$$F_1(0) = \frac{2mP(0)\gamma_z}{a_\alpha(0)} \tag{6.25a}$$

$$F_2(0) = 2 .$$

Applying eqs. (6.25) to the expression for I_{10} gives

$$I_{10} = \frac{2e^{-i\gamma_r}}{\gamma_r} \left[-1 + \frac{imP(0)}{a_\alpha(0)} \left(\frac{\gamma_z}{\gamma_r}\right)\right] .$$

We note that for γ_z/γ_r small, this reduces (to the first order) to the form

$$I_{10} \simeq \frac{-2e^{-i\gamma_r}}{\gamma_r} = \frac{-2e^{-ik_0r}}{k_0r} . \tag{6.26}$$

The c_0 term is seen to be the predominant one. We designate the contribution to the Green's function due to the I_{10} term as G'_{10}, and apply eq. (6.26) to eq. (6.10):

$$G'_{10} = \frac{H(\omega)k_0}{4\pi} I_{10} = - \frac{H(\omega)}{4\pi r} 2e^{-ik_0r} . \tag{6.27}$$

Equation (6.27), when combined with the expressions for G_∞ and G_I in eqs. (4.18) and (6.3) for γ_z/γ_r small give an interesting result. On expanding G_∞ and G_I for $\gamma_z/\gamma_r << 1$, we have

$$G_\infty \doteq \frac{H(\omega)}{4\pi} \frac{e^{-ik_0r}}{r} \tag{6.28}$$

$$G_I \simeq \frac{H(\omega)}{4\pi} \frac{e^{-ik_0r}}{r} .$$

Combining eqs. (6.28) with (6.27) gives

$$G'_{10} + G_\infty + G_I = 0 \tag{6.29}$$

for $\gamma_z/\gamma_r \ll 1$, or $(z_> + z_<)/r \ll 1$. This result shows that the source and image term plus the contribution from I_1 cancel to the first order in γ_z/γ_r. Ewing, et al. [9] discuss this phenomenon for two liquid layers. They interpret the effect as the cancelling of the direct wave by the reflected wave at grazing incidence; i.e., the reflection coefficient is -1. This is analogous to the limiting case of the Lloyd mirror effect in optics. One should note that the cancellation occurs only to the first order. Physically, the higher order terms arise as the reflection coefficient deviates from its value of -1 at grazing incidence.

To develop I_α in an asymptotic series, we apply the high-frequency approximation (6.17) to the exact expression for I_α [eq. (6.15)]. Using the asymptotic form for the Hankel function allows us to write for I_α:

$$I_\alpha = \left(\frac{2}{\pi\gamma_r}\right)^{\frac{1}{2}} e^{i\pi/4}\, e^{-i\gamma_r\alpha} \int_0^\infty e^{-\gamma_r s} s^{\frac{1}{2}} (b_0 + b_1 s + b_2 s^2 + \ldots) ds, \tag{6.30}$$

where

$$b_0 + b_1 s + b_2 s^2 + \ldots = (\alpha - is)^{\frac{1}{2}} (2i\alpha - s)^{\frac{1}{2}} \frac{e^{-i(z^2-1)^{\frac{1}{2}}\gamma_z}}{(z^2-1)^{\frac{1}{2}}} G_1(s)$$

and $z = (\alpha - is)$. The expansion about s=0 is valid only for the near-bottom case (as for the expansion of I_1) or

$$\gamma_z < \gamma_r. \tag{6.20}$$

We formally integrate eq. (6.30) term-by-term to express I_α as a series with gamma functions

$$I_\alpha \cong \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\pi/4}\, e^{-i\gamma_r\alpha} \sum_{n=0}^{\infty} b_n \frac{\Gamma(n+3/2)}{\gamma_r(n+2)} . \tag{6.31}$$

The leading term varies as γ_r^{-2}, or $(k_0r)^{-2}$ similar to the result in the preceding chapter [eq. (5.37)]. We recall that the term $\gamma_r\alpha$ in the exponential factor may be written

$\gamma_r\alpha = k_\alpha r$, so that I_α term represents a wave associated with the longitudinal wave in the subbottom. The leading term in eq. (6.31) may be calculated explicitly as follows:

$$I_{\alpha 0} = 2^{-\frac{1}{2}}\, e^{-i\gamma_r\alpha} \frac{b_0}{\gamma_r^2} e^{i\pi/4} \tag{6.32}$$

The term b_0 in the expansion is obtained from eq. (6.30) and (6.15)

$$b_0 = \alpha 2^{\frac{1}{2}}\, i^{\frac{1}{2}} \frac{e^{-(\alpha^2-1)^{\frac{1}{2}}\gamma_z}}{(\alpha^2-1)^{\frac{1}{2}}} G_1(0), \tag{6.33}$$

where

$$G_1(0) = \frac{-2i}{m(1-\alpha^2)^{\frac{1}{2}}\left(\frac{2\alpha^2}{\beta^2} - 1\right)^2}$$

Combining eqs. (6.32) and (6.33) gives for $I_{\alpha 0}$:

$$I_{\alpha 0} = \frac{-2i\alpha}{(1-\alpha^2)\gamma_r^{\ 2}\ m\left(2\frac{\alpha^2}{\beta^2} -1\right)^2}\ e^{-i(1-\alpha^2)^{\frac{1}{2}}\gamma_z}\ e^{-i\gamma_r\alpha} \tag{6.34}$$

The corresponding contribution to the Green's function is obtained from eq. (6.10):

$$G'_{\alpha 0} = \frac{k_0 H(\omega)}{4\pi} I_{\alpha 0}. \tag{6.35}$$

We write the Green's function contribution as follows:

$$G'_{\alpha 0} = \frac{H(\omega)}{4\pi} \frac{-2i\alpha}{k_0 r^2 m\left(\frac{2\alpha^2}{\beta^2} - 1\right)^2} e^{-i[(k_0^2-k_\alpha^2)^{\frac{1}{2}}(z_>+z_<)+k_\alpha r]} \tag{6.36}$$

This branch line integral represents a wave traversing the path shown in Figure 7. The wave spreads with an r^{-2} factor. The exponential factor introduces attenuation for complex k_α. To further discuss the nature of this wave, we consider the no damping case (k_α and k_β real). Recalling that $k_0 = \omega/c_0$, we write eq. (6.36) in the following form:

$$G'_{\alpha 0} = \frac{H(\omega)}{4\pi} \frac{1}{(i\omega)} \frac{2\alpha c_0}{r^2 m\left(\frac{2\alpha^2}{\beta^2} - 1\right)^2} e^{-ik_0[(\cos\theta_c)(z_> + z_<) + (\sin\theta_c) r]}, \qquad (6.36a)$$

where

$$\sin\theta_c = \alpha = c_0/c_\alpha .$$

The angle θ_c denotes the so-called critical angle of incidence. The path of the wave as shown in Figure 7 is more apparent upon writing the exponential in this form. The wave is referred to as the "refraction arrival" since the wave traverses laterally at the speed c_α along the surface of the solid bottom. This wave is the first to arrive at the receiver, since $c_\alpha > c_0$; i.e., for a "fast bottom". Ewing, et al. [9] discussed a similar wave occurring for the two liquid case. The $(i\omega)^{-1}$ factor in eq. (6.31a) implies that the refraction arrival is dispersive in the sense that the pulse shape h(t) is distorted at the receiver. The form of the frequency dependence $[(i\omega)^{-1}]$ represents an integration in the time domain, so this wave exhibits the "tail" seen in two-dimensional wave propagation. We close the discussion of the refraction arrival by writing eq. (6.36a) in its dimensional form

$$G'_{\alpha 0} = \frac{-H(\omega)}{4\pi} \frac{2ik_\alpha}{r^2 m} \frac{1}{(k_0^2 - k_\alpha^2)\left(\frac{2k_\alpha^2}{k_\beta^2} - 1\right)^2} e^{-i[(k_0^2 - k_\alpha^2)^{1/2}(z_> + z_<) + k_\alpha r]} . \qquad (6.36b)$$

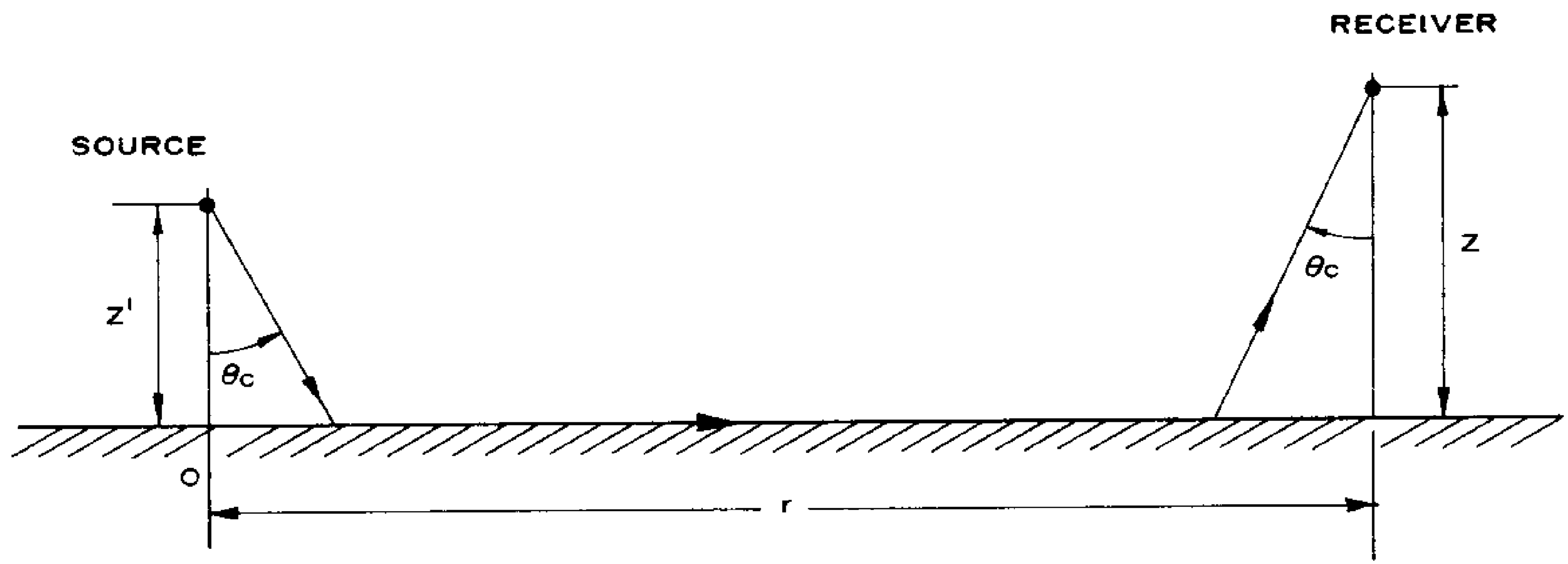


FIGURE 7

Path of the Refracted Wave

This result agrees with Honda and Nakamura's [15] result [their eq. (17)] for a liquid over an elastic halfspace. Our result differs in that a general time dependence for the input pulse is included in the $H(\omega)$ factor, and damping is implicitly included by taking k_α and k_β as complex.

The integral I_β is developed in a similar manner. We write I_β from eq. (6.16):

$$I_\beta = \left(\frac{2}{\pi}\right)^{1/2} \gamma_r^{-1/2}\, e^{-i\gamma_r}\, e^{i\pi/4} \int_0^\infty s^{1/2}\, e^{-\gamma_r s} (d_0 + d_1 s + d_2 s^2 + \ldots) ds, \qquad (6.37)$$

where

$$d_0 + d_1 s + d_2 s^2 + \ldots = \frac{e^{-(z^2-1)^{1/2}\gamma_z}}{(z^2-1)^{1/2}} (z^2-\alpha^2)^{1/2} (\beta - is)^{1/2} (2\beta i - s)^{1/2} H_1(s)$$

and $z = (\beta - is)$. The expansion of the integrand has a finite radius of convergence, so the result of the formal term-by-term integration of eq. (6.37) will be another asymptotic expansion in descending powers of γ_r. The integration gives us the following series:

$$I_\beta \simeq \left(\frac{2}{\pi}\right)^{1/2} e^{-i\gamma_r\beta}\, e^{i\pi/4} \sum_{n=0}^{\infty} d_n \frac{\Gamma(n+3/2)}{\gamma_r^{(n+2)}} \qquad (6.38)$$

This result is identical in form to eq. (6.31). The exponential term governing the propagation of this wave may be written

$e^{-i\gamma_r\beta} = e^{-irk\beta}$, so one sees that this wave is associated with the shear-wave in the subbottom. We compute the leading term in eq. (6.38) as follows:

$$I_{\beta 0} = 2^{-1/2}\, e^{i\pi/4}\, \frac{e^{-i\gamma_r\beta}}{\gamma_r^{\,2}}\, d_0, \qquad (6.39)$$

where

$$d_0 = \frac{e^{-(\beta^2-1)^{1/2}\gamma_z}}{(\beta^2-1)^{1/2}} (\beta^2-\alpha^2)^{1/2}\, 2^{1/2}\, \beta i^{1/2}\, H_1(0)$$

and

$$H_1(0) = \frac{-8m(\beta^2-1)^{\frac{1}{2}}(\beta^2-\alpha^2)^{\frac{1}{2}}}{\beta^2[m(\beta^2-1)^{\frac{1}{2}}+(\beta^2-\alpha^2)^{\frac{1}{2}}]^2} .$$

Simplifying the above gives

$$I_{\beta 0} = \frac{-8i(\beta^2-\alpha^2)m}{\gamma_r{}^2\beta[m(\beta^2-1)^{\frac{1}{2}}+(\beta^2-\alpha^2)^{\frac{1}{2}}]^2} e^{-\gamma_z(\beta^2-1)^{\frac{1}{2}}} e^{-i\gamma_r\beta} . \tag{6.40}$$

The Green's function contribution is written from eq. (6.10) as follows:

$$G'_{\beta 0} = \frac{H(\omega)}{4\pi} \frac{8m(\beta^2-\alpha^2)e^{-\gamma_z(\beta^2-1)^{\frac{1}{2}}} e^{-i\gamma_r\beta}}{k_\beta r^2[m(\beta^2-1)^{\frac{1}{2}}+(\beta^2-\alpha^2)^{\frac{1}{2}}]^2} . \tag{6.41}$$

The no damping case for $G'_{\beta 0}$ is written (recalling that $k_\beta = \omega/c_\beta$):

$$G'_{\beta 0} = \frac{H(\omega)}{4\pi} \frac{8c_\beta m(\beta^2-\alpha^2)e^{-\gamma_z(\beta^2-1)^{\frac{1}{2}}} e^{-ik_\beta r}}{(i\omega)r^2[m(\beta^2-1)^{\frac{1}{2}}+(\beta^2-\alpha^2)^{\frac{1}{2}}]^2} . \tag{6.41a}$$

The shear velocity c_β is less than c_0, so the quantity $(\beta^2-1)^{\frac{1}{2}}$ is positive and real. This indicates that exponential attenuation occurs in the vertical (z) direction due to the first exponential term. The wave propagates radially with the shear-wave speed c_β with an r^{-2} dependence. The $(i\omega)^{-1}$ factor implies that this contribution to the response integrates the pulse shape in the time domain. We note that the subbottom is "slow" with respect to the shear-wave velocity ($c_\beta < c_0$). This rules out a refraction arrival path similar to Figure 7. Instead, one has a wave bound to the liquid-solid interface radiating laterally similar to the Stoneley wave.

We write eq. (6.41) in dimensional form as follows:

$$G'_{\beta 0} = \frac{-H(\omega)}{4\pi} \frac{8im(k_\beta^2-k_\alpha^2)\ e^{-(z_>+z_<)(k_\beta^2-k_0^2)^{\frac{1}{2}}}}{r^2k_\beta[m(k_\beta^2-k_0^2)^{\frac{1}{2}}+(k_\beta^2-k_\alpha^2)^{\frac{1}{2}}]^2} e^{-ik_\beta r} . \tag{6.41b}$$

This result agrees with Honda and Nakamura's equation (21) [15], except for the more general time dependence. Again, the effect of damping is implicit in our result due to the complex nature of k_α and k_β.

D. Summary for Low Incidence Case

We summarize the results for the high-frequency, low incidence (near-bottom) case by writing the first-order response G' from eqs. (6.10), (6.11) and (6.26):

$$G'(\gamma_r,\gamma_z,\omega) \approx \frac{k_0 H(\omega)}{4\pi} \times$$

$$\times \left[2\pi i\ H_0^{(2)}(\gamma_r z_p)c_p + I_{10} + I_{\alpha 0} + I_{\beta 0} + \cdots\right], \tag{6.42}$$

where

$$I_{10} = \frac{-2e^{-ik_0 r}}{k_0 r} .$$

The first term is the Stoneley wave, and the three terms I_{10}, $I_{\alpha 0}$ and $I_{\beta 0}$ are leading terms in asymptotic expansions for branch cut integrals. The net response G is written for the low-incidence case by applying eq. (6.42) to eqs. (6.1) and (6.2)

$$G = G_\infty + G_I + G' =$$

$$= \frac{k_0 H(\omega)}{4\pi}\ [2\pi i\ H_0^{(2)}(\gamma_r z_p)c_p + I_{\alpha 0} + I_{\beta 0} + \cdots\ ], \tag{6.43}$$

where the G_∞ and G_I terms cancel with the I_1 term (to the first order) from eq. (6.29).

The net response, then, is composed of three types of wave: a surface wave (Stoneley wave) given by eq. (6.11), a refracted wave [eq. (6.36)], and another surface-type wave given by eq. (6.41).

E. Steepest-Descent Integration

The results of the preceding section [eqs. (6.42) and (6.43)] are applicable for low incidence angles or near-bottom testing where the ratio γ_z/γ_r is small. The results can be extended to moderate values of the γ_z/γ_r ratio if higher-order terms are computed for I_1, I_α and I_β as indicated in eqs. (6.22), (6.31) and (6.38). Computation of higher-order terms requires considerable algebraic manipulation where complicated expressions must be expanded in power series about s=0, as indicated in eqs. (6.21), (6.30) and (6.37). The expansions have finite radii of convergence, so all the integrals give divergent series in descending powers of the large parameter γ_r. This type of series is an asymptotic expansion, as has been discussed earlier. Usually, the first few terms of the series gives accurate results for sufficiently large values of the dominant large parameter. At some point in the expansion, the terms start getting large, causing the series to diverge. The series is usually truncated just before the terms start diverging.

The algebraic difficulty in computing higher-order terms for the branch line integrals, combined with the uncertainty of convergence of the series prompts us to evaluate the Green's function G' given in eq. (6.6) using another approach. We apply the method of steepest descents [4], [20] to evaluate the Green's function. This method entails deforming the original path of integration in the complex plane in such a way that the integrand is significant for only a small region in the new path of integration. To apply the method to the integral given in eq. (6.6), we first express the Green's function in the complex plane by substituting z=x+iy for x and denote the deformed contour as Γ_s:

$$G'(\gamma_r,\gamma_z,\omega) =$$

$$= \frac{-H(\omega)k_0}{4\pi}\int_{\Gamma_s} \frac{e^{-(z^2-1)^{\frac{1}{2}}\gamma_z}}{(z^2-1)^{\frac{1}{2}}} \frac{(z^2-\alpha^2)^{\frac{1}{2}}}{D_1(z^2)} H_0^{(2)}(\gamma_r z)z\,dz, \tag{6.44}$$

where

$$D_1(z^2) = m(z^2-1)^{\frac{1}{2}} \left[\left(\frac{2z^2}{\beta^2} - 1\right)^2 - \frac{4z^2}{\beta^4}(z^2-\alpha^2)^{\frac{1}{2}}(z^2-\beta^2)^{\frac{1}{2}}\right] + (z^2-\alpha^2)^{\frac{1}{2}}.$$

The path Γ_s is as yet unspecified. We now apply the high-frequency assumption by using the asymptotic expression for the Hankel function [see eq. (5.17)]:

$$H_0^{(2)}(\gamma_r z) = \left(\frac{2}{\pi\gamma_r z}\right)^{\frac{1}{2}} e^{i\pi/4}\, e^{-i\gamma_r z}\ , \tag{6.45}$$

$$|\gamma_r z| >> 1.$$

We may apply eq. (6.45) provided the path Γ_s does not lie near the origin where $|z|$ is small. Substituting eq. (6.45) into (6.44) gives

$$G'(\gamma_r,\gamma_z,\omega) =$$

$$= \frac{-H(\omega)k_0}{2/\pi}\left(\frac{2}{\pi\gamma_r}\right)^{\frac{1}{2}} e^{i\pi/4}\int_{\Gamma_s} \frac{(z^2-\alpha^2)^{\frac{1}{2}}}{(z^2-1)^{\frac{1}{2}}} \frac{z^{\frac{1}{2}}}{D_1(z^2)}\, e^{-i[\gamma_r z+(1-z^2)^{\frac{1}{2}}\gamma_z]}dz \tag{6.46}$$

We introduce the angle of incidence θ by defining, from Figure 5:

$$\tan\theta = \frac{r}{(z_>+z_<)} = \frac{\gamma_r}{\gamma_z}\ . \tag{6.47}$$

We may write for r and $(z_>+z_<)$ the following:

$$r = R_I \sin\theta,$$

$$(z_>+z_<) = R_I \cos\theta\ , \tag{6.47a}$$

where R_I is given in eq. (6.3) as

$$R_I = [(z_>+z_<)^2 + r^2]^{\frac{1}{2}}.$$

We write eq. (6.46) using these results as follows:

$$G'(\gamma_r,\gamma_z,\omega) =$$

$$= \frac{-H(\omega)k_0}{4\pi}\left(\frac{2}{\pi\gamma_r}\right)^{\frac{1}{2}}\int_{\Gamma_s}\frac{(z^2-\alpha^2)^{\frac{1}{2}}}{(z^2-1)^{\frac{1}{2}}}\frac{z^{\frac{1}{2}}}{D_1(z^2)}e^{-ik_0R_I[z\sin\theta+(1-z^2)^{\frac{1}{2}}\cos\theta]}dz \tag{6.48}$$

The parameter k_0R_I may be written as γ_I, a ratio of the path length of the reflected wave to the wavelength. We write eq. (6.48) in the form

$$G'(\gamma_I,\theta,\omega) = \frac{-H(\omega)k_0}{4\pi}\left(\frac{2}{\pi\gamma_r}\right)^{\frac{1}{2}} I_s, \tag{6.49}$$

where

$$I_s = \int_{\Gamma_s}\frac{(z^2-\alpha^2)^{\frac{1}{2}}}{(z^2-1)^{\frac{1}{2}}}\frac{z^{\frac{1}{2}}}{D_1(z^2)}e^{-\gamma_I f(z)}$$

and $f(z) = i[z\sin\theta + (1-z^2)^{\frac{1}{2}}\cos\theta]$.

We assume now that the parameter γ_I is large, or

$$\gamma_I >> 1. \tag{6.50}$$

This defines a radiation zone in the liquid field. To determine the steepest-descent path Γ_s we compute the point of stationarity of $f(z)$, the factor in the exponential of the integral I_s. This is defined by

$$f'(z)\Big|_{z=z_0} = 0. \tag{6.51}$$

The point of stationarity z_0 is then given by the relation

$$\tan\theta = \frac{z_0}{(1-z_0^2)^{\frac{1}{2}}} \tag{6.51a}$$

We may also write, from eq. (6.51a):

$$\sin\theta = z_0$$

$$\text{and}\qquad \cos\theta = (1-z_0^2)^{\frac{1}{2}}. \tag{6.51b}$$

We now expand f(z) about the point z_0 as follows:

$$f(z) = f(z_0) + f''(z_0)\,\frac{1}{2!}(z-z_0)^2 + \ldots\ , \tag{6.52}$$

where

$$f(z_0) = i$$

and $$f''(z_0) = -i[\cos\theta]^{-2}.$$

We write f(z) to the second order as

$$f(z) \approx i - \frac{i}{\cos^2\theta}\,\frac{(z-z_0)^2}{2} \tag{6.52a}$$

Applying this result to eq. (6.49) for I_s gives

$$I_s \approx e^{-i\gamma_I}\int_{\Gamma_s} \frac{(z^2-\alpha^2)^{\frac{1}{2}}}{(z^2-1)^{\frac{1}{2}}}\,\frac{z^{\frac{1}{2}}}{D_1(z^2)}\,e^{i\gamma_I\frac{(z-z_0)^2}{2\cos^2\theta}}\,dz \tag{6.53}$$

Now, the path Γ_s is defined near z_0 by transforming the exponential factor $i\gamma_I\,\frac{(z-z_0)^2}{2\cos^2\theta}$ to a real negative quantity. For the path Γ_s near z_0, we write

$$(z-z_0) = re^{i\alpha}, \tag{6.54}$$

where α must be $\pi/4$. A similar path is used by Landau and Lifschitz [25] for two liquids. The integral I_s then becomes approximately

$$I_s \approx e^{-i\gamma_I}\,\frac{(z_0^2-\alpha^2)^{\frac{1}{2}}}{(z_0^2-1)^{\frac{1}{2}}}\,\frac{z_0^{\frac{1}{2}}}{D_1(z_0^2)}\int_{-\infty}^{\infty} e^{-r^2\left(\frac{\gamma_I}{2\cos^2\theta}\right)}\,dr$$

$$= e^{-i\gamma_I}\,\frac{(z_0^2-\alpha^2)^{\frac{1}{2}}}{(z_0^2-1)^{\frac{1}{2}}}\,\frac{z_0^{\frac{1}{2}}}{D_1(z_0^2)}\left(\frac{2\pi}{\gamma_I}\right)^{\frac{1}{2}}\cos\theta, \tag{6.55}$$

where we have used the result [37]

$$\int_{-\infty}^{\infty} e^{-a^2x^2}\, dx = \left(\frac{\pi}{a^2}\right)^{\frac{1}{2}}.$$

We write the approximate Green's function from eqs. (6.55) and (6.49) as follows:

$$G'(\gamma_I,\theta,\omega) \approx \frac{-H(\omega)}{4\pi}\,\frac{2}{R_I}\, e^{-i\gamma_I}\left[\frac{(z_0^2-\alpha^2)^{\frac{1}{2}}}{D_1(z_0^2)}\right], \tag{6.56}$$

where we have used the relations (6.51b).

The result is given a more useful form by adding the image source term G_I to G'. From eqs. (6.2) and (6.3), we have

$$G_1 = G_I + G' \approx \frac{H(\omega)}{4\pi}\,\frac{e^{-ik_0R_I}}{R_I}\left[\frac{N_1(z_0^2)}{D_1(z_0^2)}\right], \tag{6.57}$$

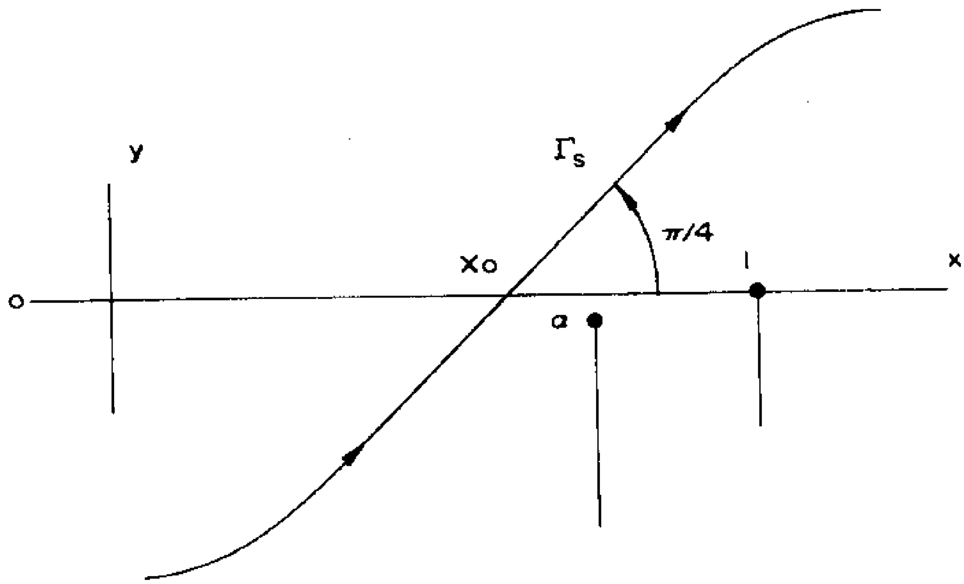
where

$$N_1(z_0^2) = D_1(z_0^2) - 2(z_0^2 - \alpha^2)^{\frac{1}{2}}.$$

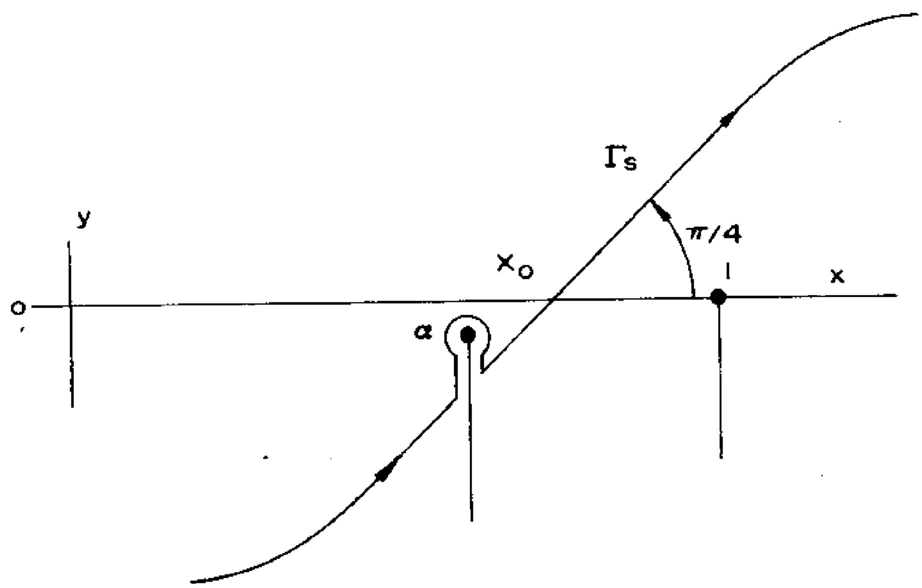
Equation (6.57) is the expression for the reflected wave. It has the spherical spreading factor R_I^{-1}, and propagates along the path shown in Figure 5 at the speed c_0. The quantity in brackets $\{N_1/D_1\}$ represents the plane-wave reflection coefficient for an acoustic wave reflecting off a solid halfspace. The reflection coefficient is a function of the angle of incidence, as we recall from eq. (6.51b):

$$z_0 = \sin\theta. \tag{6.51b}$$

The result (6.55) applies for a limited range of the angle of incidence. Referring to Figure 8 and eq. (6.51b), we see that the point of stationarity z_0 varies with the angle of incidence θ. The deformed contour Γ_s appears as in Figure 8a for low angles of



a) No Refracted Wave ($\theta<\theta_c$)



b) With Refracted Wave ($\theta>\theta_c$)

FIGURE 8

Steepest Descent Paths: Radiation Zone

incidence. If the angle of incidence increases beyond a critical angle θ_c, the path Γ_s must loop around the branch cut for $z=\alpha$, as shown in Figure 8b. The critical angle θ_c is given by the relation

$$z_0 = \alpha = \sin\theta_c \tag{6.58}$$

for no damping, or

$$\sin\theta_c = \frac{k_\alpha}{k_0} = \frac{c_0}{c_\alpha} . \tag{6.58a}$$

When damping occurs, the critical angle increases because the point $z=\alpha$ lies off the real axis. For small damping, the critical angle occurs when

$$\sin\theta_c = \frac{k_\alpha^R + |k_\alpha^I|}{k_0} , \tag{6.58b}$$

since the path Γ_s is inclined 45^o from the real axis near z_0. The result given by eq. (6.56) is valid only for $\theta < \theta_c$, where θ_c is given by eq. (6.58b).

For angles of incidence greater than θ_c, one must include the contribution to the integral due to the loop around the branch cut for $z=\alpha$. This loop integral has already been calculated to the first order as $G'_{\alpha 0}$ in eq. (6.36). We recall that this term represents the refraction arrival. The result $G'_{\alpha 0}$ is valid provided the angle of incidence is well away from the critical angle. The loop integral has finite limits, as can be seen from Figure 8b. If θ approaches θ_c, the finite limits must be taken into account.

We may write for the response G' the following:

$$G'(\gamma_I,\theta,\omega) \approx -\frac{H(\omega)}{4\pi}\frac{2}{R_I}e^{-i\gamma_I}\left[\frac{(z_0^2-\alpha^2)^{\frac{1}{2}}}{D_1(z_0^2)}\right] + G'_{\alpha 0} \tag{6.59}$$

where G'_0 is given by eq. (6.36), and

$$\theta > \theta_c.$$

Combining the G' response with the image source term G_I gives

$$G_1 = G_I + G' \simeq \frac{H(\omega)}{4\pi} \frac{e^{-ik_0R_I}}{R_I} \left[\frac{N_1(z_0^2)}{D_1(z_0^2)}\right] + G'_{\alpha 0} \tag{6.60}$$

for $\theta > \theta_c$. This result is similar to eq. (6.57) for $\theta < \theta_c$, except that the refracted wave term $G'_{\alpha 0}$ has been added due to the loop around the branch cut.

From eq. (6.1), we see the response in the sum of the direct wave G_∞ and G_1, or

$$G = G_\infty + G_1. \tag{6.1}$$

The net response in the radiation zone for moderate angles of incidence is composed of the direct wave G_∞, eq. (4.18), a reflected wave given as the right-hand side of eq. (6.56) and the refraction arrival $G'_{\alpha 0}$ given by eq. (6.36). The refraction arrival occurs only for

$$\theta > \theta_c,$$

where θ_c is given in eq. (6.58b).

VII. SUMMARY AND CONCLUSIONS

A. Summary

Expressions for the acoustic response in the frequency domain have been obtained for the n-layer viscoelastic halfspace. The response is expressed in integral form. The n = 1 case (a homogeneous solid halfspace) has been integrated for both finite and infinite depth of the overlying liquid layer.

A discussion of the problem and a brief survey of related work is presented in Chapter I. The nature of the sedimentary subbottom in shallow water is summarized in Table 1 (from Hamilton's data). The effect of damping processes in the subbottom is discussed, with the result that the bottom may be considered an elastic solid with superimposed damping (Voigt viscoelastic model).

The anlaysis of the problem starts in Chapter II. The conservation laws and an entropy production inequality are presented as the governing equations for the media. A discussion of the linearization process is then given. The linearization is based on small disturbances (wave fronts) superimposed on a uniform equilibrium or ambient state.

The constitutive equations are developed for the viscoelastic solid undergoing small deformations using an energy approach. The equation of motion governing the mechanical field is given as eq. (2.50). Following this, the equation of motion for the inviscid fluid is shown to be a special case of the elastic (undamped) solid. A wave equation (2.57) is derived using an equation of state for the pressure. The acoustic wave propagation speed is shown to be related to an isentropic elastic modulus and to a thermodynamic derivative.

The vector field equations (2.50) and (2.51) are simplified in Chapter III. First the discussion is limited to the elastic field equation (no damping). The displacement is decomposed into longitudinal (curl-less) and transverse (divergence-less) parts. This separates the equation into two wave operators for each polarization. A Fourier transform in time is then introduced which reduces the wave operators to Helmholtz operators. Then solutions to the homogeneous forms (no source terms) are developed using a scalar potential function for the longitudinal field and a vector potential for the transverse field. The discussion parallels closely the classical electrodynamic wave propagation problem [32]. One obtains for cylindrical coordinates expressions for three polarizations: one longitudinal and two transverse [eq. (3.20)]. Each polarization is expressed in terms of a scalar function satisfying a scalar Helmholtz operator.

The viscoelastic medium is then discussed. After applying the Fourier transform, one finds that the field equations reduce to a form identical to the elastic solid. Each polarization is governed by a Helmholtz operator [eqs. (3.24) and (3.25)], where the wavenumbers are complex quantities instead of real due to the presence of damping terms.

Homogeneous forms are used for the governing equations in the solid. The field excitation occurs in the overlying liquid layer, so an inhomogeneous form (with source term) must apply in the inviscid liquid. The source term in the liquid is taken as a localized, longitudinal disturbance representing an acoustic transducer. The disturbance is modeled as a point source with an arbitrary time dependence for the strength. The acoustic response to the source is then the system Green's function, which is governed by eq. (3.36).

The stress and displacement fields for the cylindrical coordinate system are developed in terms of the scalar potential functions for each polarization. These expressions are used in the sequel to evaluate the boundary conditions at each interface between media.

In Chapter IV the solution to the boundary-value problem for arbitrary layers in the subbottom is developed. The response is obtained from eq. (3.36) using a formal Green's function treatment [50], [14] where a Fourier-Bessel transform is applied. The transformation reduces the governing equation to an ordinary differential equation in one dimension. The Green's function for the unbounded fluid is obtained by matching boundary conditions at the liquid-solid interface.

The potentials in the solid are written in a form (4.22) compatible with the Green's function. Boundary conditions are applied at an arbitrary solid-solid interface. These boundary conditions are then expressed as a recurrence relation (4.30a). Successive applications of the recurrence relation enable one to express the potentials in the first solid layer in terms of those in the last [eq. (4.31)]. Then the acoustic field is matched to the first solid layer to obtain eq. (4.33). The acoustic potential is obtained as a Green's function using the recurrence relation. The transformed solution for arbitrary layers is given in eq. (4.37). Special cases of eq. (4.38) are obtained for a single solid layer, an infinite liquid layer depth and a combination of both in eqs. (4.40), (4.41) and (4.42), respectively.

In Chapter V the acoustic response is obtained for the first special case: the single solid layer, a halfspace. An integral form is obtained by taking an inverse Fourier-Bessel transform. The integral form is manipulated into a form more convenient for contour integration by introducing a Hankel function in place of the Bessel function and changing the limits of integration. The expression is then written in nondimensional form in eq. (5.8). After a discussion of the branch and pole singularities, a contour is selected and the residue theorem is applied. The Green's function is then expressed in eq. (5.14a) as a sum of residues and line integrals around each branch cut. The significance of the residue terms is discussed and each branch line integral is written as a definite integral in exact form. The branch line integrals are evaluated approximately by considering the high-frequency far-field case. Each integral is expressed as an asymptotic series in descending powers of a large non-dimensional parameter. Two of the branch line integralsvanish due to the nature of the integrand.

The final result is given in eq. (5.42), where one has three expressions: a residue series and two asymptotic series for the branch line integrals. The residue series has many terms due to the large number of poles of the integrand. The poles are frequency-dependent, making the residue series highly dispersive. The discussion is concluded by observing that the response is too complicated for further analysis for the present problem. The large number of residue terms and the required root search make computations too cumbersome. In addition, the expression for the response is only valid in the far-field due to assumptions made in the evaluation of the branch line integrals.

In Chapter VI, the case of the semi-infinite liquid over the homogeneous solid halfspace is developed. The Green's function is written as the sum of a direct wave, an image source term and a residual term in eq. (6.4). The direct wave and image source terms are readily obtained from Sommerfeld's results [46], [49]. We integrate the residual term using the same procedure as in the preceding chapter. The result is given in eq. (6.10) as the sum of three branch line integrals and a residue term. The residue term represents a damped Stoneley wave [8]. In a discussion assuming small damping, the Stoneley wave is shown to propagate laterally along the liquid-solid interface with a cylindrical spreading law. The effect of the wave decays exponentially as the distance from the interface.

Expressions for the branch line integrals are given in eqs. (6.14), (6.15) and (6.16). These are expressed as asymptotic expansions for the high-frequency case in eqs. (6.23), (6.31) and (6.38). These waves are interpreted as reflected and refracted waves after computing the first term in each series. The first term of eq. (6.23) is given in eq. (6.26). A discussion of this wave shows that it cancels to the first order with the direct and image source terms, corresponding to cancellation of the direct wave by the reflected wave at grazing incidence. The branch-line integral (6.23), when combined with the image source term, is then interpreted as the reflected wave. The other two branch line integrals are discussed after computing the first-order terms in each expansion. The first-order contributions to the Green's functions are given in expressions (6.36) and (6.41). The first contribution is interpreted as a refracted wave traveling along a path shown in Figure 7. The second is a modified refracted wave propagating along the interface and decaying exponentially as the distance from the interface.

These results are restricted to grazing angles of incidence due to a convergence condition on the branch line integrals [eq. (6.20)]. An expression for the response is obtained for moderate angles of incidence by applying the method of steepest descent. The original path of integration along the real axis is replaced with the one shown in Figure 8. One obtains two results for the Green's function; eqs. (6.57) and (6.60). The first is valid for angles of incidence less than a critical angle defined in eq. (6.58b), and the second for angles larger than the critical angle.

One sees that the response is the sum of the direct wave, a reflected wave and a refracted wave which appears only for angles of incidence greater than the critical. The steepest-descent result is valid for moderate angles of incidence not too near the critical angle in the radiation zone or high-frequency regime.

B. Results and Conclusions

The primary results of this study are the expressions for the acoustic response given in Chapters V and VI for the liquid layer and liquid halfspace, respectively. The predominant response in the liquid layer was found to consist of a residue series, each term representing a mode of propagation. This result was shown to be inconvenient for modeling acoustic sounding. We then developed in Chapter VI the response for infinite depth of the liquid layer, a case of interest when water surface reflections are not important. After a discussion of the significance of the high-frequency far-field approximation, the response was shown to consist of the sum of several types of waves, each of which was associated with a singularity in the complex plane. Each type of wave was interpreted physically from the algebraic results by analyzing the leading term in its asymptotic series representation.

Two cases were considered: near-bottom grazing incidence and moderate angles of incidence. In the first case the reflected wave was shown to cancel with the direct wave (to the first order). Two types of refracted wave occurred. One was associated with the compressional or longitudinal wave in the subbottom, traveling in a path indicated in Figure 7. The second one was associated with the shear or transverse wave in the subbottom. This was found to be a wave propagating along the interface, decaying exponentially in the vertical direction. Another wave, the Stoneley wave, occurred due to a pole singularity. This was an interface wave that spread laterally like a cylindrical wave.

For moderate angles of incidence, the steepest-descent method of integration was applied. Here the response consisted of the direct wave, a reflected wave and a refracted wave appearing only for angles of incidence greater than a critical angle.

These results provide physical insight into the subbottom identification problem discussed in Chapter I. The insight is especially useful for designing acoustic sounding experiments and for analyzing data. The grazing-incidence results show that one may directly obtain information on the compressional wave propagation in the subbottom by observing the "first arrival time" associated with the refracted wave traveling the path shown in Figure 7. This technique, called "refraction shooting", is commonly used in offshore petroleum prospecting. The shear-wave propagation in the subbottom can be determined indirectly by observing the damped Stoneley wave. This technique has been used by Hamilton [11].

His model for the Stoneley wave, described in Reference [2], does not systematically take into account the three-dimensional nature of the wave or the effect of damping. These effects were explicitly included in the present study. We conclude, then, that near-bottom grazing incidence testing can yield information directly or indirectly on the wave propagation (including damping) in the subbottom.

Oblique incidence sounding can also yield information on the subbottom. The feasibility of this approach has been demonstrated by Breslau [1], as mentioned in Chapter I. The results obtained here show that more refined information may be obtained through inclusion of damping and use of a range of incidence angles. (Breslau was concerned with only normal incidence.)

The results of Chapter VI can then be used to design arrays of acoustic transducers and for analysis of data. Specifically, computer studies may be performed using numerical data from Hamilton's results (Table 1). Since consistent damping data is not available, one must infer the effect from <u>in situ</u> data.

In Chapter IV the acoustic response was expressed in integral form for a subbottom with an arbitrary number of parallel layers. Due to algebraic complexity, this integral form was evaluated only for the single-layer case in the succeeding chapters. The general result, however, is new and provides a means for investigating sub-layering effects. Two approaches can be used: direct computer studies in which the integration is performed numerically, and a combined computer-analytical study in which the integral form is expressed as a series of residue and branch line integrals. The second approach is a generalization of Chapters V and VI. The algebraic complexity of the integrand for more than one layer in the subbottom precludes the direct calculation of the residue term and the branch line integrals. However, the calculations may be performed on the computer quite readily.

Some insight on the sub-layering problem was obtained by Jardetzsky [16]. He found that the only branch line integrals contributing to the response were those for the singularities in the last layer, the halfspace. The branch singularities in the intermediate layers did not contribute due to the form of the integrand. The response for the more general case will then consist of two branch line integrals and a residue series. The response may be obtained by an appropriate root search using numerical techniques and by computing numerically the two branch line integrals. The steepest descent method may also be applied to the multi-layer problem for the infinite water depth case. The integrand is more complex, but the procedure parallels closely the development in Chapter VI.

In Chapter III the field excitation occurring in the liquid was taken, after a physical discussion, as a point monopole source. A Green's function formalism was introduced conveniently since the acoustic response due to the point source can be considered to be the system's Green's function. The formalism was applied systematically to the multi-layer problem in Chapter IV. This problem is ideal for application of the Green's function formalism due to the type of field excitation, the shorthand notation and the systematic nature of the computational procedure. This study appears to be the first where the Green's function formalism was applied to multi-layer problems. The geophysics literature (cf. Reference [5]) develops the response by computing acoustic potentials separately above and below the source in the liquid. The field excitation in the liquid is taken into account by adding a source term, the form of which is obtained from Sommerfeld's [46] result. The Green's function formalism combines the two expressions for the potentials above and below the source using reciprocity. In addition, the field excitation is taken into account by the use of an inhomogeneous form (point source) for the acoustic field equation.

C. Recommendations

Further development of the results presented is required for solving the subbottom identification problem. The immediate work must combine numerical or computer studies with experimental results. Some general conclusions must be arrived at concerning the nature of subbottom damping. In addition, the effects of sub-layering must be determined. This study provides the necessary models for interpreting experimental data obtained to determine these effects.

The following areas of study are recommended:

1. *In situ* acoustic sounding at grazing angles of incidence (near-bottom testing) using the results of Chapter VI as a model for interpretation of test data. Predicted pulse shapes may be obtained for representative bottom types using Hamilton's results and Fourier synthesis. Comparison of test data with predicted pulse shapes may be made using coring data for the site.

2. Computer analysis of the effects of sublayering using the results of Chapter IV, coring data for representative sites and Hamilton's results.

3. Computation of higher-order terms in the expansions in Chapter VI. Also further analysis of behavior of the integral form for

mathematically pathological cases where singularities lie close together in complex plane using the approach of Van Der Waerden [56]. The case where the angle of incidence approaches the critical angle is of particular interest.

4. Further analytical study of the mechanism of damping for unconsolidated sediments combined with an experimental program in the laboratory. The experimental program might be performed in the ultrasonic frequency regime to reduce the size of the experiment.

Further development of the modeling is highly dependent upon experimental results, as guidance is needed to determine the direction of further analysis. More specific recommendations cannot be made without further insight from carefully designed and executed experiments carried out in the field.

Other factors not considered in this study may be significant. These include random inhomogenieties in the media, non-parallel and non-planar layering structure and thermal gradients in the liquid layer. Analysis of these effects should be performed if experimental results indicate that any of these are important.

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APPENDIX A

ISOTROPIC TENSORS

In deriving constitutive relations for media having isotropic physical properties, it is necessary to develop expressions for isotropic tensors of the second and fourth order. For convenience in computation we restrict the development to Cartesian tensor notation.

Isotropy implies that the quantity in question has properties invariant with respect to the orientation of the coordinate system. One writes the representation for a vector (first order tensor) upon rotation (change of orientation) as follows:

$$u_j' = a_{ij}u_i, \tag{A.1}$$

where the prime denotes the representation in the rotated frame, and the a_{ij} are the cosines of the angles between the original $i^{\underline{th}}$ axis and the new $j^{\underline{th}}$ axis. The inverse transformation of eq. (A.1) is

$$u_i = a_{ij}u_k' \tag{A.1a}$$

Applying eq. (A.1a) to (A.1) gives

$$u_j' = a_{ij}a_{ik}u_k' \tag{A.2}$$

Equation (A.2) implies that

$$a_{ij}a_{ik} = \delta_{jk},$$

where δ_{jk} is the Kronecker delta. Furthermore, $a_{ji}a_{ki} = \delta_{jk}$. One may write the coordinates of the rotated frame [from eq. (A.1)] as

$$x_j' = a_{ij}x_i \ .$$

We see that

$$\frac{\partial x_j'}{\partial x_i} = a_{ij} \ .$$

From the inverse transformation, $\frac{\partial x_i}{\partial x_j'} = a_{ij}$.

We introduce a tensor of the second order ω_{ik} by requiring it to transform according to the following law

$$\omega'_{jl} = a_{ij}a_{kl}\omega_{ik}. \tag{A.3}$$

The condition of istropy for the second-order tensor is:

$$\omega'_{ij} = \omega_{ij} \tag{A.4}$$

Say we write

$$\omega_{ij} = c\delta_{ij} , \tag{A.5}$$

where c is a scalar.

The transformed form is

$$\omega'_{ij} = c\delta'_{ij} \tag{A.5a}$$

If ω_{ij} is isotropic, eq. (A.4) must hold, or

$$\delta'_{ij} = \delta_{ij} \tag{A.6}$$

from eqs. (A.5). We write, since δ_{ij} is evidently a second-order tensor

$$\delta'_{ij} = a_{ki}a_{lj}\delta_{kl} = a_{ki}a_{kj} = \delta_{ij}$$

This result shows eq. (A.6) is satisfied, so the representation of eq. (A.5) is an isotropic tensor. An example of an isotropic second-order tensor is the stress tensor for an inviscid fluid (eq. 2.47):

$$\sigma_{ij} = -p'\delta_{ij}.$$

One requires isotropic fourth-order tensors in the constitutive relations for linear solids having material isotropy [see eqs. (2.25) and (2.38)]. We have a relation of the form:

$$\sigma_{ij} = E_{ijmn}\varepsilon_{mn}. \tag{A.7}$$

In the rotated (primed) frame one has

$$\sigma'_{op} = E'_{opqr}E'_{qr}. \tag{A.8}$$

To show that E_{ijmn} is a fourth-order tensor we write

$$\sigma_{ij} = a_{io}\, a_{jp}\, a'_{op} \tag{A.9a}$$

and

$$\varepsilon'_{qr} = a_{mq}\, a_{nr}\, \varepsilon'_{mn} \tag{A.9b}$$

Substituting eq. (A.9b) into eq. (A.8) gives

$$\sigma'_{op} = a_{mq}\, a_{nr}\, E'_{opqr}\, \varepsilon_{mn} \tag{A.10}$$

If one multiplies both sides of eq. (A.10) by $a_{io}a_{jp}$ one has, from eq. (A.9a)

$$\sigma_{ij} = a_{io}\, a_{jp}\, a_{mq}\, a_{mr}\, E'_{opqr}\, \varepsilon_{mn} \tag{A.11}$$

One sees from eqs. (A.11) and (A.7) that

$$E_{ijmn} = a_{io}\, a_{jp}\, a_{mq}\, a_{nr}\, E'_{opqr}, \tag{A.12}$$

is the transformation for a fourth-order tensor, as can be seen from a generalization of the transformation of eq. (A.1a). The condition of material isotropy for the relations (A.7) and (A.8) is, from eq. (A.12)

$$E_{ijmn} = E'_{ijmn} = a_{io}\, a_{jp}\, a_{mq}\, a_{nr}\, E'_{opqr} \tag{A.13}$$

Equation (A.13) is satisfied for three products of Kronecker delta functions:

$$E_{ijmn} \sim \begin{cases} \delta_{ij}\, \delta_{mn}, \\ \delta_{im}\, \delta_{jn}, \\ \delta_{in}\, \delta_{jm}, \end{cases} \tag{A.14}$$

as can be verified by direct substitution. If one has the following symmetries:

$$E_{ijmn} = E_{jimn} = E_{ijnm} = E_{mnij}, \tag{A.15}$$

the most general isotropic fourth-order tensor can be constructed from a linear combination of the three factors in eq. (A.14) (See Refs. [18] and [19]):

$$E_{ijmn} = \lambda\delta_{ij}\,\delta_{mn} + \mu(\delta_{im}\,\delta_{jn} + \delta_{in}\,\delta_{jm}). \tag{A.16}$$

Equation (A.16) is the form used for constitutive relations in eqs. (2.25) and (2.38). The symmetries in eq. (A.15) arise due the symmetry of the stress and strain (or strain-rate) tensors and due to thermodynamic considerations.

APPENDIX B

THE ROOTS OF THE FREQUENCY EQUATION: FINITE DEPTH OF LIQUID LAYER

We wish to find solutions to eq. (5.11), the frequency equation for the finite liquid layer over the homogeneous viscoelastic halfspace. Solutions represent poles of the integrand in eq. (5.14), which in turn are associated with residue terms representing modal behavior in the acoustic response. We write the frequency equation as

$$D(z_i^2) = 0, \tag{5.11}$$

where $\pm$ 3i is the i^{th} solution in the complex (z) plane and from eq. (5.8):

$$D(z^2) = m(z^2-1)^{\frac{1}{2}}[2z^2-\beta^2)^2 - 4(z^2-\alpha^2)^{\frac{1}{2}}(z^2-\beta^2)^{\frac{1}{2}}z^2]\cosh[k_0h_0(z^2-1)^{\frac{1}{2}}] + \\ +\beta^4(z^2-\alpha^2)^{\frac{1}{2}}\sinh[k_0h_0(z^2-1)^{\frac{1}{2}}].$$

As mentioned in Chapter V, the roots of the frequency equation for the liquid over an elastic solid were investigated by Schermann[42]. His study applies to eq. (5.11) if α and β are taken as positive real numbers. Viscoelasticity of the Voigt type makes α and β complex numbers appearing in the fourth quadrant. This is evident if one recalls the definition of α and β :

$$\alpha = \frac{k_\alpha}{k_0}$$

and

$$\beta = \frac{k_\beta}{k_0} \quad , \tag{5.7}$$

where k_α and k_β are given in eqs. (3.24) and (3.25) and k_α is real and positive.

The damping is relatively small, as mentioned in Chapter I. This implies that we may write α and β in the form:

$$\alpha = \alpha_0 - i\varepsilon\alpha'$$

and (B.1)

$$\beta = \beta_0 - i\varepsilon\beta'$$

where α' and β' are real and positive and

$$\varepsilon << 1 \tag{B.2}$$

We may formally apply perturbation theory to the problem, setting the solution, z, in the form

$$z = z_0 - i\varepsilon z' \tag{B.3}$$

The term z_0 corresponds to an undamped solution to equation (5.11), as can be shown by expanding eq.(5.11) about the undamped equation state:

$$D(z,\alpha,\beta) = 0 = D(z_0,\alpha_0,\beta_0) + -i\varepsilon\left\{ \frac{\partial D}{\partial z} z' + \frac{\partial D}{\partial \alpha}\alpha' + \frac{\partial D}{\partial \beta}\beta' \right\}\Bigg|_{z_0,\alpha_0,\beta_0} + \cdots \tag{B.4}$$

We set each term associated with a given power of ε to zero in (B.4), giving as a result to the first order:

$$D(z_0,\alpha_0,\beta_0) = 0 \tag{B.5}$$

and

$$z' = -\left[\frac{\frac{\partial D}{\partial \alpha}\alpha' + \frac{\partial D}{\partial \beta}\,'}{\frac{\partial D}{\partial z}} \right]_{z_0,\alpha_0,\beta_0} \tag{B.6}$$

One sees from this result that the solution z_0, to the undamped equation is the zeroth order part of the solution for the damped equation. The first order part of the solution (z') is given in terms of the derivatives of the undamped equation.

Since a first-order solution to eq. (5.11) is also a solution to eq. (B.5) for the undamped case, Schermann's results are applicable. We recall that Schermann found a finite number of real roots. From the discussion in Ewing, et al. [7], the real roots were found to lie in the region

$$\alpha_0 < x < 1 \tag{B.7}$$

for two liquids. The same result can be shown to apply for the solid bottom if the shear wave velocity is small. We note from Table 1 that

$$c_\beta \ll c_\alpha \quad \text{or} \quad \beta_0 \gg 1$$

We rewrite eq. (5.11) for a real root $z_0 = x$ as follows:

$$\frac{m(x^2-1)^{\frac{1}{2}}}{(x^2-\alpha_0^2)^{\frac{1}{2}}} \left\{ \left[2\left(\frac{x}{\beta_0}\right)^2 - 1 \right]^2 - \frac{4i}{\beta_0^4}(x^2-\alpha_0^2)^{\frac{1}{2}} \quad (\beta_0^2-x^2)^{\frac{1}{2}} \right] = \tag{B.8}$$

$$-\tanh \quad [k_o h_0 (x^2-1)^{\frac{1}{2}}]$$

For β very large, eq. (B.8) reduces to

$$\frac{m(x^2-1)^{\frac{1}{2}}}{(x^2-\ {}_0^2)^{\frac{1}{2}}} = -\tanh\ [k_0 h_0 (x^2-1)^{\frac{1}{2}}] \quad , \tag{B.9}$$

the frequency equation for two liquids [7]. This can have real solutions, x, only if eq. (B.7) holds. We rewrite eq. (B.9) as follows:

$$\frac{m(1-x^2)^{\frac{1}{2}}}{(x^2-\alpha^2)^{\frac{1}{2}}} = -\tan\ [k_0 h_0 (1-x^2)^{\frac{1}{2}}] \quad , \tag{B.9a}$$

where all the square roots are positive real numbers. We note that solutions to equation (B.9a) are frequency-dependent due to the $k_0 h_0$ factors on the right-hand side. One may graphically determine the solutions to eq. (B.9a) as was done in reference [7] in Figure 4-4. The result is that N non-trivial real roots appear in the region given in eq. (B.7), where N is the largest integer satisfying the inequality

$$\left(\frac{2N+1}{2}\right)\pi \leq k_0 h_0 (1-\alpha^2)^{\frac{1}{2}} \tag{B.10}$$

Introducing the wavelength $\lambda_0 = 2\pi/k_0$ into eq. (B.10) gives:

$$\left(\frac{2N+1}{4}\right)\pi \leq \left(\frac{h_0}{\lambda_0}\right)(1-\alpha^2)^{\frac{1}{2}}$$

We see that for high frequencies corresponding to large values of the ratio h_0/λ_0, one has many solutions N.

One sees from the form of eq. (B.8) that the number of poles, N, is the same for the liquid over the elastic solid to the first order for $\beta \gg 1$. This can be shown formally by expanding about the two liquid case ($\beta_0 = 0$). The zeroth order solution will be given by eq. (B.9a). The effect of the small rigidity shifts each pole slightly in the complex plane.

As a result, we see that solutions to eq. (5.11) for small damping and β_0 small lie close to the real axis and slightly in the fourth quadrant. We write eq. (B.3) in the form

$$z = x_0 - i\varepsilon z' \;, \tag{B.11}$$

where $\alpha_0 < x_0 < 1$

and $\varepsilon << 1$

The number N of roots is given by eq. (B. 10).

These results have been verified for a typical example of marine sediments. Complex roots of eq. (5.11) were determined numerically using Hamilton's data for fine sand (see Table 1). A small amount of damping was assumed for α and β. The frequency was taken as 3500 Hz and the water depth h_0 was 30 meters. Other parameters were:

$$c_0 = 1501 \text{ m/sec}$$

$$c_\alpha = 1742 \text{ m/sec}$$

$$c_\beta = 382 \text{ m/sec}$$

$$\rho_0 = 1.025 \text{ g/cm}^3$$

$$\rho_1 = 1.98 \text{ g/cm}^3$$

The number N of roots found was 70, a value that agreed with eq. (B.10). The complex roots were found to lie very close to the real axis, and the real parts fall in the range given by eq. (B.7). These results verified the observations based on the perturbational argument given.

APPENDIX C

THE ROOTS OF THE FREQUENCY EQUATION: INFINITE DEPTH OF LIQUID LAYER

We wish to solve eq. (6.8):

$$D_1(z^2) = 0$$

where, from eq. (6.6):

$$D_1(z^2) = m(z^2-1)^{1/2}\left[\left(\frac{2z^2}{\beta^2}-1\right)^2 - \frac{4z^2}{\beta^4}(z^2-\alpha^2)^{1/2}(z^2-\beta^2)^{1/2}\right] + (z^2-\alpha^2)^{1/2}$$

We apply the small damping assumption to eq. (6.8) and use a perturbation as was done in Appendix B. On applying eqs. (B.1), (B.2) and (B.3) and expanding D_1 about the undamped state, we obtain

$$D_1(z,\alpha,\beta) = 0 = D_1(z_0,\alpha_0,\beta_0) +$$

$$-i\varepsilon\left[\frac{\partial D_1}{\partial z}z' + \frac{\partial D_1}{\partial\alpha}\alpha' + \frac{\partial D_1}{\partial\beta}\beta'\right]_{z_0,\alpha_0,\beta_0} + \ldots \tag{C-1}$$

Setting each order of ε to zero in eq. (C-1 gives

$$D_1(z_0,\alpha_0,\beta_0) = 0 \tag{C-2}$$

$$z = \left\{\frac{\dfrac{\partial D_1}{\partial\alpha}\alpha' + \dfrac{\partial D_1}{\partial\beta}\beta'}{\dfrac{\partial D_1}{\partial z}}\right\}_{z_0,\alpha_0,\beta_0} \tag{C-3}$$

As mentioned in Chapter VI, solutions to eq. (C-2) (the undamped equation) were obtained by Strick and Ginsbarg [52]. For marine sediments, where

$$\alpha_0 < 1 < \beta_0 \quad .$$

A real root x_p of eq. (C-2) exists, where

$$x_p > \beta_0 \quad . \tag{C-4}$$

One may write the complex root z_p to the first order as

$$z_p = x_p - i\varepsilon z' \ , \tag{C-5}$$

where z' is given by eq. (C-3). The real part x_p is obtained from the curves in Reference [52].

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